Asymptotic Pseudomodes of Toeplitz Matrices

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Questions in probability and statistical physics lead to the problem of finding the eigenvectors associated with the extreme eigenvalues of Toeplitz matrices generated by Fisher-Hartwig symbols. We here simplify the problem and consider pseudomodes instead of eigenvectors. This replacement allows us to treat fairly general symbols, which are far beyond Fisher-Hartwig symbols. Our main result delivers a variety of concrete unit vectors $x_n$ such that if $T_n(a)$ is the $n \times n$ truncation of the infinite Toeplitz matrix generated by a function $a \in L^1$ satisfying mild additional conditions and $\lambda$ is in the range of this function, then $\|T_n(a)x_n - \lambda x_n\| \to 0$.

Keywords: Toeplitz matrix, Fisher-Hartwig symbol, eigenvector, pseudomode, fractional Brownian motion

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1 Introduction and Main Results

The $n \times n$ Toeplitz matrix $T_n(a)$ generated by a complex-valued function $a$ belonging to $L^1 := L^1(0, 2\pi)$ is the matrix $(a_{j-k})_{j,k=1}^n$ constituted by the Fourier coefficients

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} a(\theta) e^{-ik\theta} d\theta \quad (k \in \mathbb{Z}),$$

of the function $a$. For a real number number $\alpha \in (0, \frac{1}{2})$, put

$$\omega_\alpha(\theta) = |1 - e^{i\theta}|^{-2\alpha} = 2^{-2\alpha} \left| \sin \frac{\theta}{2} \right|^{-2\alpha}. $$

This function, which is a special so-called Fisher-Hartwig symbol, is in $L^1$ and its Fourier coefficients are

$$(\omega_\alpha)_k = \Gamma(1 - 2\alpha) \frac{\sin \pi \alpha}{\pi (|k| + \alpha)} \frac{\Gamma(|k| + 1 + \alpha)}{\Gamma(|k| + 1 - \alpha)} \sim \Gamma(1 - 2\alpha) \frac{\sin \pi \alpha}{|k|^{1-2\alpha}}.$$
where \( x_k \sim y_k \) means that \( x_k/y_k \to 1 \). Clearly, \( \omega_\alpha \) is real-valued, even (after extension to a \( 2\pi \)-periodic function on \( \mathbb{R} \)), and \( \min_\theta \omega_\alpha(\theta) = \omega_\alpha(\pi) = 2^{-2\alpha} \). The matrices \( T_n(\omega_\alpha) \) are symmetric and positive definite. Let

\[
\lambda_1(T_n(\omega_\alpha)) \leq \lambda_2(T_n(\omega_\alpha)) \leq \ldots \leq \lambda_n(T_n(\omega_\alpha))
\]

be the eigenvalues of \( T_n(\omega_\alpha) \). It is well known that \( \lambda_k(T_n(\omega_\alpha)) \to \omega_\alpha(\pi) \) as \( n \to \infty \) for each fixed \( k \geq 1 \). Matlab shows that the normalized eigenvectors for \( \lambda_k(T_n(\omega_\alpha)) \) are very close to

\[
\sqrt{\frac{2}{n+1}} \left( (-1)^{j+1} \sin \frac{jk\pi}{n+1} \right)^n_{j=1}
\]

(see Figures 1 to 3).

![Figure 1: We see a normalized eigenvector for \( \lambda_1(T_{30}(\omega_{1/4})) = 0.7074 \) (crosses) and the values of \( \frac{1}{\sqrt{40}} (-1)^{j+1} \sin \frac{sj}{40} \) for \( j = 1, \ldots, 39 \) (circles).](image)

The matrices \( T_n(\omega_\alpha) \) are of interest for probabilists and statistical physicists. Let us first explain the connection with probability theory. Fractional Brownian motion (FBM for short) with Hurst index \( H \in (0,1) \) is by definition the centered Gaussian process \( B^H_t, t > 0 \), with covariance function \( \mathbb{E}[B^H_t B^H_s] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) \). It has \( (H-\varepsilon) \)-Hölder continuous trajectories for any \( \varepsilon > 0 \), so the process gets more and more irregular as \( H \to 0 \). The case \( H = \frac{1}{2} \) corresponds to Brownian motion, which is Markov. Leaving aside this case, one has a non-Markovian Gaussian process with stationary increments, with very different properties depending on whether \( H > \frac{1}{2} \) or \( H < \frac{1}{2} \). A result known as the “invariance principle” (see the book by Samorodnitsky...
and Taqqu [15], Theorem 7.2.11) states that, if $H > \frac{1}{2}$, then

$$N^{-H} \sum_{1 \leq j \leq [N]} Y_j \rightarrow \frac{1}{\sqrt{H|2H-1|}} B^H_t \quad (0 \leq t \leq 1)$$

as $N \to \infty$.

(convergence of finite-dimensional distributions) if $Y_j, j \in \mathbb{Z}$, is any stationary sequence of centered Gaussian variables with a covariance function such that

$$\mathbb{E}[Y_j Y_k] = r(|j-k|) \rightarrow |k-j|^{2H-2} \text{ as } |j-k| \to \infty.$$

Hence one may choose in particular the stationary covariance function associated with the above Toeplitz matrix $T(\omega_\alpha)$ for $\alpha = H - \frac{1}{2}$, namely, $\mathbb{E}[Y_j Y_k] = (\omega_{H-\frac{1}{2}})_{j-k}$. The same is true for $H < \frac{1}{2}$ provided that $\sum_{j \in \mathbb{Z}} \mathbb{E}[Y_0 Y_j] = \sum_{j \in \mathbb{Z}} r(|j|) = 0$, which is valid for $T_N(\omega_\alpha)$ (note that $\alpha \in (-\frac{1}{2}, 0)$ is negative in that case, which leads to a bounded Toeplitz operator). Knowing a quasi-exact diagonalization of the covariance matrix may help to compute the law of some functionals of FBM.

As for physicists, they are interested in studying finite-size effects for Gaussian lattice models with long-range interactions. Namely, consider real-valued spins $\sigma(i), i \in \Lambda$ on a $d$-dimensional lattice $\Lambda \subset \mathbb{Z}^d$, and attach to each configuration $\{\sigma\} = (\sigma(i))_{i \in \Lambda} \in \mathbb{R}^\Lambda$ a Boltzmann weight proportional to $\exp -\beta Q(\{\sigma\})$, where $\beta > 0$ is the inverse of the temperature and $Q$ is a quadratic form with a spectrum which is bounded below. The lattice is here considered to be infinite in $d-1$ dimensions, and finite with $N$ layers in the $d$th direction. Assuming $d = 1$, this is equivalent to the above discretization of FBM if one sets $Q_N = (T_N(\omega_\alpha))^{-1}$ on $\Lambda = \{1, \ldots, N\}$. The matrix $Q_N$ is no longer a Toeplitz matrix, but $(Q_N)_{i,j} \sim C|i-j|^{-1-2\alpha}$ as $N$ and $|i-j|$...
Figure 3: These are a normalized eigenvector for $\lambda_2(T_{39}(\omega_{1/4})) = 0.7082$ (crosses) and the values of $\left(-1\right)^j \sin \frac{2\pi j}{40}$ for $j = 1, \ldots, 39$ (circles).

go to infinity with $i, j$ staying close to the middle, that is, with $i/N, j/N \to \frac{1}{2}$; this is a consequence of an exact formula for $Q_N$ which known as the Duduchava-Roch theorem; see [7], Prop. 2.2. Alternatively, one may set $Q_N = T_N(\omega - \alpha)$, which gives an interaction depending only on the distance of the sites, but then of course the covariance matrix is no more stationary. In any case, physicists have been considering a Gaussian variant of this model (called “ferromagnetic spherical model” in the literature, see [3]) for $\alpha \in (0, 1/2)$, exhibiting a second-order phase transition at a positive critical temperature. Fine computations of finite-size effects have been obtained (see the book by Brankov, Danchev, and Tonchev [2]) for the partition function (free energy), the susceptibility (related to the integrated correlation function), the shift of the critical temperature, etc., relying on the non too physical “periodic boundary condition”, which is more or less equivalent to replacing the Toeplitz matrix with its optimal circulant approximation. For the simplest computations, only the spectrum of the Toeplitz matrix is needed, so their results might be extended to the case of free boundary conditions (corresponding to the usual Toeplitz matrix) by taking into account corrections to Szegö’s theorem on the asymptotic spectrum. But for the most interesting results, one needs to diagonalize the quadratic form $Q$. Note that different (including free) boundary conditions have been analyzed in the case of short-range interactions, where the Toeplitz matrix has only a finite number of non-zero diagonals and can be easily diagonalized; see [1]. Even in that simple case, finite-size effects depend strongly on the choice of boundary conditions.

This paper arose from the attempt to prove that (1) is indeed close to a $k$th eigenvector of $T_n(\omega_{\alpha})$. We have not been able to achieve this goal, but in the course
of our efforts we gained some insights that might be of independent interest.

Let \( \{A_n\}_{n=1}^{\infty} \) be a sequence of \( d(n) \times d(n) \) matrices. We think of \( A_n \) as a linear operator on \( \mathbb{C}^{d(n)} \) with the \( \ell^2 \) norm. The operator norm (= spectral norm) of \( A_n \) is denoted by \( \|A_n\| \). Fix a point \( \lambda \in \mathbb{C} \). We call a sequence \( \{x_n\}_{n=1}^{\infty} \) of nonzero vectors \( x_n \in \mathbb{C}^{d(n)} \) an asymptotic eigenvector for \( \lambda \) if there exist two sequences \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{v_n\}_{n=1}^{\infty} \) such that

\[
v_n \neq 0, \quad A_n v_n = \lambda_n v_n, \quad \lambda_n \to \lambda,
\]

and we refer to the sequence \( \{x_n\}_{n=1}^{\infty} \) as an asymptotic pseudomode for \( \lambda \) if

\[
\left\| \frac{A_n x_n - \lambda x_n}{\|x_n\|} \right\| \to 0.
\]

Frequently we simply say that \( x_n \) itself is an asymptotic eigenvector or an asymptotic pseudomode. Trefethen and Embree’s book \cite{18} is the standard reference to this topic.

If \( \|A_n\| \leq M < \infty \) for all \( n \), then

\[
\left\| \frac{A_n x_n - \lambda x_n}{\|x_n\|} \right\| \leq M \left\| \frac{x_n}{\|x_n\|} - \frac{v_n}{\|v_n\|} \right\| + |\lambda| \left\| \frac{x_n}{\|x_n\|} - \frac{v_n}{\|v_n\|} \right\|
\]

and hence asymptotic eigenvectors are automatically asymptotic pseudomodes. This is no longer true if \( \limsup \|A_n\| = \infty \) (see Proposition 2.1 below). Furthermore, independently of whether \( \|A_n\| \) remains bounded or not, asymptotic pseudomodes need not to be asymptotic eigenvectors (Theorems 1.1 and 1.2 provide us with plenty of examples). Since \( \|T_n(\omega_\alpha)\| \sim C_\alpha n^{2\alpha} \) with some constant \( C_\alpha \) (see \cite{8}), it follows that for \( T_n(\omega_\alpha) \) \( (0 < \alpha < 1/2) \) the notions of asymptotic eigenvectors and asymptotic pseudomodes are two completely different concepts: an asymptotic eigenvector is not necessarily an asymptotic pseudomode and vice versa.

By a trigonometric polynomial we understand a function \( f \) of the form \( f(x) = \sum_{k=-m}^{m} c_k e^{ikx} \). In the following two theorems \( f \) is supposed to be not identically zero.

**Theorem 1.1** Let \( a \in L^1 \), suppose \( \|T_n(a)\| = O(n^{2\alpha}) \) with \( 0 \leq \alpha < 1/2 \), and assume \( a \) is Lipschitz continuous in a neighborhood of \( \pi \). If \( f \) is an arbitrary trigonometric polynomial, then

\[
\left( \sqrt{\frac{2}{n+1}} (-1)^{j+1} f \left( \frac{j\pi}{n+1} \right) \right)^n_{j=1}
\]

is an asymptotic pseudomode of \( T_n(a) \) for \( \lambda = a(\pi) \).

This theorem implies that the vectors (1) are asymptotic pseudomodes of \( T_n(\omega_\alpha) \) for \( \lambda = \omega_\alpha(\pi) \). As \( \omega_\alpha \) is symmetric about \( \pi \) and \( \|T_n(\omega_\alpha)\| \sim C_\alpha n^{2\alpha} \), part (b) of the following theorem delivers lots of pseudomodes of \( T_n(\omega_\alpha) \) for \( \lambda = a(\pi) \) beyond trigonometric polynomials.
Theorem 1.2 Let $a \in L^1$ be Lipschitz continuous in a neighborhood of $\pi$ and suppose at least one of the following two conditions is satisfied: (a) $a \in L^\infty$ and $f \in C[0,\pi]$, (b) $a(\pi - \theta) = a(\pi + \theta)$ for all sufficiently small $\theta$, $\|T_n(a)\| = O(n^{2\alpha})$ with $\alpha < 1/2$, and $f \in C^3[0,\pi]$. Then

$$\left(\sqrt{\frac{2}{n+1}}(-1)^{j+1}f\left(\frac{j\pi}{n+1}\right)\right)_{j=1}^n$$

is an asymptotic pseudomode of $T_n(a)$ for $\lambda = a(\pi)$.

Theorems 1.1 and 1.2 concern individual pseudomodes. A result on the collective behavior of asymptotic pseudomodes is in [20]. Let $U_n = \frac{1}{\sqrt{n}}\left(e^{2\pi ijk/n}\right)_{j,k=0}^{n-1}$ be the Fourier matrix and denote by $U_ne_k$ the $k$th column of $U_n$. Zamarashkin and Tyrtyshnikov [20] observed that if $a \in L^2$, then

$$\sum_{k=0}^{n-1} \|T_n(a)U_ne_k - \lambda_{n-k+1}(C_n(a))U_ne_k\|^2 = o(n) \quad (2)$$

where $\text{diag}(\lambda_0(C_n(a)), \ldots, \lambda_{n-1}(C_n(a))) := U_nC_n(a)U_n^*$ and $C_n(a)$ is the optimal circulant matrix for $T_n(a)$, that is, the uniquely determined circulant matrix $X$ for which the Frobenius norm of $T_n(a) - X$ is minimal. They also stated that (2) does not necessarily hold for $a \in L^1$, but that if $a \in L^1$, then for each $\varepsilon > 0$ the number of $k \in \{0,1,\ldots, n-1\}$ for which

$$\min_{\lambda} \|T_n(a)U_ne_k - \lambda U_ne_k\| \geq \varepsilon$$

is $o(n)$. We here prove the following.

Theorem 1.3 Let $a \in L^1$ and $\|T_n(a)\| = O(n^{2\alpha})$ for some number $\alpha \in [0,1/2)$. Suppose the $2\pi$-periodic extension of $a$ is Lipschitz continuous in a neighborhood of $\theta_0 = 2\pi(1 - \beta) \in [0,2\pi]$. Then

$$\|T_n(a)U_ne_k - a(\theta_0)U_ne_k\| \to 0$$

whenever $n \to \infty$ and $k = \beta n + O(1)$. In other terms, $U_ne_{\beta n + O(1)}$ is an asymptotic pseudomode of $T_n(a)$ for $\lambda = a(2\pi(1 - \beta))$.

While (2), after division by $n$, may be regarded as a result on convergence in the mean, Theorem 1.3 may be viewed as a result on pointwise convergence.


2 Remarks

Here is the simple observation we mentioned in Section 1.

**Proposition 2.1** Let \( \{A_n\} \) be a sequence of matrices such that \( \|A_n\| \to \infty \). Suppose \( \lambda \in \mathbb{C} \) is a limiting point of the spectra of \( A_n \), that is, there exist \( \lambda_n \) and \( v_n \) such that \( \|v_n\| = 1 \), \( A_n v_n = \lambda_n v_n \), and \( \lambda_n \to \lambda \). Then the sequence \( \{A_n\} \) has asymptotic eigenvectors for \( \lambda \) which are not asymptotic pseudomodes for \( \lambda \).

**Proof.** There are \( y_n \) such that \( \|y_n\| = 1 \) and \( \|A_n y_n\| = \|A_n\| \). Put \( \varepsilon_n = 1 / \sqrt{\|A_n\|} \) and \( x_n = v_n + \varepsilon_n y_n \). Since \( \|x_n\| \to 1 \) and \( \|x_n - v_n\| = \varepsilon_n \to 0 \), the sequence \( \{x_n\} \) is an asymptotic eigenvector for \( \lambda \). On the other hand,

\[
A_n x_n - \lambda x_n = A_n (v_n + \varepsilon_n y_n) - \lambda (v_n + \varepsilon_n y_n) = \varepsilon_n A_n y_n - (\lambda - \lambda_n) v_n - \lambda \varepsilon_n y_n,
\]

whence \( \|A_n x_n - \lambda x_n\| \geq \varepsilon_n \|A_n\| - |\lambda - \lambda_n| - \lambda \varepsilon_n \to \infty \). □

The following is obvious.

**Proposition 2.2** Let \( \{A_n\} \) be a sequence of matrices and \( \lambda \in \mathbb{C} \). Suppose \( A_n - \lambda I \) is invertible for all \( n \). Then \( \| (A_n - \lambda I)^{-1} \| \geq \alpha_n \) for all \( n \) if and only if there exists a sequence \( \{x_n\} \) such that \( \|A_n x_n - \lambda x_n\|/\|x_n\| \leq 1/\alpha_n \). □

From the work of Kac, Murdock, Szegö [10], Widom [19], Parter [11], [12], and Serra [16], [17] it is known that there exist constants \( c_1, c_2 \in (0, \infty) \) such that

\[
\frac{c_1}{n^2} \leq \lambda_1(T_n(\omega_\alpha)) - \omega_\alpha(\pi) \leq \frac{c_2}{n^2}
\]

for all \( n \). Since \( T_n(\omega_\alpha) \) is selfadjoint, this implies that

\[
\frac{n^2}{c_2} \leq \| (T_n(\omega_\alpha) - \omega_\alpha(\pi)I)^{-1} \| \leq \frac{n^2}{c_1}.
\]

Thus, Proposition 2.2 reveals that there exist asymptotic pseudomodes \( x_n \) such that

\[
\frac{\|T_n(\omega_\alpha)x_n - \omega_\alpha(\pi)x_n\|}{\|x_n\|} \leq \frac{c_2}{n^2}
\]

(3)

does not exceed \( c_2/n^2 \), but that there is no asymptotic pseudomode \( x_n \) for which (3) is \( o(1/n^2) \).

For a vector \( x = (x_0, \ldots, x_{n-1}) \in \mathbb{C}^n \), we define the trigonometric polynomial \( Fx \) by

\[
(Fx)(\theta) = \sum_{\ell=0}^{n-1} x_\ell e^{i\ell \theta} \quad (\theta \in \mathbb{R}).
\]
Clearly, the $j$th component of $T_n(a)x$ equals the $j$th Fourier coefficient of the product of $a$ and $Fx$,

\[(T_n(a)x)_j = \frac{1}{2\pi} \int_0^{2\pi} a(\theta)(Fx)(\theta)e^{-ij\theta} d\theta \quad (j = 0, \ldots, n - 1). \quad (4)\]

The following proposition is a simple application of (4) and reveals that $T_n(a)$ has asymptotic pseudomodes that are completely different from those of Theorems 1.1 and 1.2.

**Proposition 2.3** Let $a \in L^1$ and suppose $|a(\theta) - a(\pi)|$ is $O(|\theta - \pi|)$ near $\pi$. Then the vectors $x_n$ given by

\[(x_n)_j = (-1)^j \binom{n - 1}{j} \quad (j = 0, \ldots, n - 1)\]

are an asymptotic pseudomode of $T_n(a)$ for $\lambda = a(\pi)$.

**Proof.** We have

\[(Fx_n)(\theta) = (1 - e^{i\theta})^{n - 1} = \left(-2i\sin\frac{\theta}{2}\right)^{n - 1} e^{i\theta(n - 1)/2}.

This implies that

\[\|x_n\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| 2\sin\frac{\theta}{2}\right|^{2n-2} d\theta = \frac{\Gamma(2n-2)}{\sqrt{\pi} \Gamma(n)} \sim \frac{2^{2n-2}}{\sqrt{\pi n}},\]

which, incidentally, can also be obtained from

\[\|x_n\|^2 = \sum_{j=0}^{n-1} \binom{n - 1}{j}^2 = \binom{2n - 2}{n - 1} \sim \frac{2^{2n-2}}{\sqrt{\pi n}}.

Formula (4) yields

\[\delta_j := \frac{(T_n(a)x_n - a(\pi)x_n)_j}{\|x_n\|} = \frac{1}{2\pi\|x_n\|} \int_0^{2\pi} [a(\theta) - a(\pi)](Fx_n)(\theta)e^{-ij\theta} d\theta.

Thus,

\[|\delta_j| \leq C_1 \frac{n^{1/4}}{2^n} \int_0^{2\pi} |a(\theta) - a(\pi)| \sin\frac{\theta}{2} d\theta.\]

By assumption, there are a $\mu \in (0, \pi/2)$ and a finite constant $K$ such that

\[|a(\theta) - a(\pi)| \leq K|\theta - \pi| \leq C_2 K \left|\cos\frac{\theta}{2}\right|\]

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for $\theta \in (\pi - \mu, \pi + \mu)$, whence

$$n^{1/4} \int_{|\theta - \pi| < \mu} |a(\theta) - a(\pi)| \left| \sin \frac{\theta}{2} \right|^{n-1} d\theta \leq n^{1/4} \int_{|\theta - \pi| < \mu} \left| \cos \frac{\theta}{2} \right| \left| \sin \frac{\theta}{2} \right|^{n-1} d\theta$$

$$= n^{1/4} \int_{|x| < \mu} \left| \sin \frac{x}{2} \right| \left| \cos \frac{x}{2} \right|^{n-1} dx \leq 2 n^{1/4} \int_{0}^{\pi/2} \sin \frac{x}{2} \left( \cos \frac{x}{2} \right)^{n-1} dx$$

$$= -4n^{1/4} \frac{1}{n} \left( \cos \frac{x}{2} \right)^{n \pi/2} \bigg|_{0}^{\infty} = \frac{4}{n^{3/4}}.$$  

On the other hand,

$$n^{1/4} \int_{|\theta - \pi| > \mu} |a(\theta) - a(\pi)| \left| \sin \frac{\theta}{2} \right|^{n-1} d\theta \leq n^{1/4} \left( \sin \frac{\mu}{2} \right)^{n-1} \int_{|\theta - \pi| > \mu} |a(\theta) - a(\pi)| d\theta = O \left( \frac{1}{n^{3/4}} \right).$$

Consequently,

$$\|\delta\|^2 = \sum_{j=0}^{n-1} |\delta_j|^2 = O \left( \frac{1}{n^{6/4}} \right) = O \left( \frac{1}{n^{1/2}} \right) = o(1). \quad \Box$$

**Remark 2.4** Let $T(a) = (a_{j-k})_{j,k=1}^{\infty}$ be the infinite Toeplitz matrix generated by $a$. This matrix induces a bounded operator on $\ell^2 := \ell^2(\mathbb{N})$ if and only if $a \in L^\infty$. Suppose, for simplicity, $a$ is the restriction to $[0, 2\pi]$ of a continuous and $2\pi$-periodic function on $\mathbb{R}$. Then the range $\mathcal{R}(a)$ of $a$ is a closed, continuous, and naturally oriented curve in the plane. For $\lambda \in \mathbb{C} \setminus \mathcal{R}(a)$, denote by wind $(a, \lambda)$ the winding number of $\mathcal{R}(a)$ about $\lambda$. If wind $(a, \lambda) = 0$, then $\{T_n(a)\}$ does not have asymptotic pseudomodes, because then, by a classical result of Gohberg and Feldman [9], $\|(T_n(a) - \lambda I)^{-1}\| = O(1)$. Let wind $(a, \lambda) = -m < 0$. We then can write $a(x) - \lambda = b(x)e^{-imx}$ and the operator $T(b)$ can be shown to be invertible on $\ell^2$. Put

$$u_j = T^{-1}(b)e_j \quad (j = 1, \ldots, m)$$

where $e_j \in \ell^2$ is the sequence whose $j$th term is 1 and the remaining terms of which are zero. One can show that $u_1, \ldots, u_m$ form a basis in the null space of $T(a) - \lambda I$. Let finally $P_n : \ell^2 \rightarrow \mathbb{C}^n$ be projection onto the first $n$ coordinates. In [6], it was proved that a sequence $\{x_n\}$ of vectors $x_n \in \mathbb{C}^n$ is an asymptotic pseudomode of $\{T_n(a)\}$ for $\lambda$ if and only if there exist $c_{1}^{(n)}, \ldots, c_{m}^{(n)} \in \mathbb{C}$ and $z_n \in \mathbb{C}^n$ such that

$$\frac{x_n}{\|x_n\|} = c_{1}^{(n)} u_1 + \ldots + c_{m}^{(n)} u_m + z_n, \quad \sup_{n \geq 1, 1 \leq j \leq m} |c_{j}^{(n)}| < \infty, \quad \lim_{n \rightarrow \infty} \|z_n\| = 0.$$  

Paper [6] also contains a characterization of all asymptotic pseudomodes of $\{T_n(a)\}$ for points $\lambda$ with wind $(a, \lambda) = m > 0$.  

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Reichel and Trefethen [14] were probably the first to observe that if \( a \) is a trigonometric polynomial, \( \lambda \in \mathbb{C} \setminus \mathcal{R}(a) \), and wind \((a, \lambda) \neq 0\), then \( \| (T_n(a) - \lambda I)^{-1} \| \) increases exponentially fast and hence, by Proposition 2.2, there exist asymptotic pseudomodes \( x_n \) such that \( \|T_n(a)x_n - \lambda x_n\|/\|x_n\| \) decreases exponentially fast. See also Theorem 7.2 of [18]. Under the assumption that \( \lambda \in \mathbb{C} \setminus \mathcal{R}(a) \) and wind \((a, \lambda) \neq 0\), the growth of the norms \( \| (T_n(a) - \lambda I)^{-1} \| \) for piecewise continuous and general continuous functions \( a \) is studied in [4] and [5], respectively.

In connection with all these results, the contribution of this paper to the topic is that we deliver concrete pseudomodes of \( \{T_n(a)\} \) for points \( \lambda \in \mathcal{R}(a) \).

\[ \square \]

3 Preliminaries

We now start with the proof of our main results. For a Riemann integrable function \( f \) on \([0, \pi]\), we denote by \( \tilde{V}_n f \) and \( V_n f \) the vectors in \( \mathbb{C}^n \) given by

\[
\tilde{V}_n f = \left( \sqrt{\frac{2}{n+1}} \left( -1 \right)^{j+1} f \left( \frac{j\pi}{n+1} \right) \right)_j^n, \quad V_n f = \left( \sqrt{\frac{2}{n-1}} (-1)^j f \left( \frac{j\pi}{n-1} \right) \right)_j^n.
\]

The vectors \( \tilde{V}_n f \) are aesthetically more appealing, but we found that working with \( V_n f \) is more convenient. Obviously,

\[
\lim_{n \to \infty} \| \tilde{V}_n f \|^2 = \lim_{n \to \infty} \| V_n f \|^2 = \frac{2}{\pi} \int_0^\pi |f(x)|^2 dx. \tag{5}
\]

**Proposition 3.1** Let \( \{A_n\} \) be an arbitrary sequence of matrices and let \( \lambda \in \mathbb{C} \). If \( f \in C[0, \pi] \), then \( \tilde{V}_n f \) and \( V_n f \) are simultaneously asymptotic eigenvectors of \( \{A_n\} \) for \( \lambda \) or not.

**Proof.** This is clear because

\[
\left\| V_n f - \sqrt{\frac{n+1}{n-1}} \tilde{V}_n f \right\|^2 = \frac{2}{n-1} \sum_{j=0}^{n-1} \left| f \left( \frac{j\pi}{n-1} \right) - f \left( \frac{(j+1)\pi}{n+1} \right) \right|^2 = o(1) \tag{6}
\]

and

\[
\lim_{n \to \infty} \| V_n f \|^2 = \lim_{n \to \infty} \left\| \sqrt{\frac{n+1}{n-1}} \tilde{V}_n f \right\|^2 = \frac{2}{\pi} \int_0^\pi |f(x)|^2 dx. \tag{7}
\]

**Proposition 3.2** Let \( \{A_n\} \) be an arbitrary sequence of matrices satisfying \( \|A_n\| = O(n^{2\alpha}) \) (0 \( \leq \alpha < 1/2 \)) and let \( \lambda \in \mathbb{C} \). Suppose \( f \in C[0, \pi] \) if \( \alpha = 0 \) and \( f \) is Lipschitz continuous on \([0, \pi]\) if \( 0 < \alpha < 1/2 \). Then \( \tilde{V}_n f \) and \( V_n f \) are simultaneously asymptotic pseudomodes of \( \{A_n\} \) for \( \lambda \) or not.

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Proof. We abbreviate $V_n f$ and $\sqrt{\frac{n+1}{n-1}} V_n f$ to $x_n$ and $y_n$, respectively. We have

$$\|A_n x_n - \lambda x_n\| - \|A_n y_n - \lambda y_n\| \leq \|A_n - \lambda I\| \|x_n - y_n\|. \quad (8)$$

If $\|A_n\| = O(1)$ and $f$ is continuous, then (6), (7), (8) imply that $x_n$ is an asymptotic pseudomode if and only if $y_n$ has this property. If $f$ is Lipschitz continuous on $[0, 1]$, $|f(t) - f(s)| \leq K|t - s|$, then (6) is at most

$$\frac{2K^2}{n-1} \sum_{j=0}^{n-1} \frac{|2j - n + 1|}{n^2 - 1} = O\left(\frac{1}{n} \sum_{j=0}^{n-1} \left(\frac{j^2}{n^2} + \frac{1}{n^2}\right)\right) = O\left(\frac{1}{n^2}\right).$$

Thus, $\|x_n - y_n\| = O(1/n)$. From (7) and (8) we therefore see that $x_n$ and $y_n$ are simultaneously asymptotic pseudomodes or not. □

Lemma 3.3 Let $a \in L^1$ and $\|T_n(a)\| = O(n^{2\alpha}) \ (0 \leq \alpha < 1/2)$. Suppose $f \in C[0, \pi]$ if $\alpha = 0$ and $f \in C^2[0, \pi]$ if $0 < \alpha < 1/2$. If

$$\|T_{2m+1}(a)V_{2m+1} f - \lambda V_{2m+1} f\|/\|V_{2m+1} f\| \to 0 \quad \text{as} \quad m \to \infty,$$

then

$$\|T_{2m}(a)V_{2m} f - \lambda V_{2m} f\|/\|V_{2m} f\| \to 0 \quad \text{as} \quad m \to \infty.$$

Proof. We think of $\mathbb{C}^n$ as a space of columns indexed from 0 to $n - 1$. We denote by $P_n : \mathbb{C}^{n+1} \to \mathbb{C}^n$ the operator that is given by $(P_n x)_j = x_j$ for $0 \leq j \leq n - 1$. Let $n$ be even. Obviously,

$$T_n(a) P_n V_{n+1} f - \lambda P_n V_{n+1} f = P_n T_n(a) V_{n+1} f - \lambda P_n V_{n+1} f - \begin{pmatrix} a_{-n} \\ \vdots \\ a_{-1} \end{pmatrix} (V_{n+1} f)_n.$$

Let $g(x) = f(x) - f(\pi)$. Then the column $(a_{-n}, \ldots, a_{-1})^T (V_{n+1} f)_n$ equals

$$\begin{pmatrix} a_{-n} \\ \vdots \\ a_{-1} \end{pmatrix} (V_{n+1} g)_n + \begin{pmatrix} a_{-n} \\ \vdots \\ a_{-1} \end{pmatrix} \frac{(-1)^n}{\sqrt{n}} f(\pi) = \begin{pmatrix} a_{-n} \\ \vdots \\ a_{-1} \end{pmatrix} \frac{(-1)^n}{\sqrt{n}} f(\pi)$$

and the squared norm of the last column is

$$|f(\pi)|^2 \frac{|a_{-1}|^2 + \ldots + |a_{-n}|^2}{n} = o(1),$$

because $|a_{-n}|^2 \to 0$ by the Riemann-Lebesgue theorem and the arithmetic mean of a sequence converging to zero converges to zero, too. It follows that

$$\|T_n(a) P_n V_{n+1} f - \lambda P_n V_{n+1} f\|/\|V_{n+1} f\| \to 0.$$
and hence also that
\[ \sqrt{\frac{n}{n-1}} \|T_n(a)P_nV_{n+1}f - \lambda P_nV_{n+1}f\|/\|V_nf\| \to 0. \]

We have
\[ \left\| \sqrt{\frac{n}{n-1}} P_nV_{n+1}f - V_nf \right\|^2 = \frac{2}{n-1} \sum_{j=0}^{n-1} \left| f\left(\frac{j\pi}{n}\right) - f\left(\frac{j\pi}{n-1}\right) \right|^2, \]
which is \( o(1) \) if \( f \in C[0,\pi] \). Thus, in the case \( \alpha = 0 \) we conclude that
\[ \|T_n(a)V_nf - \lambda V_nf\|/\|V_nf\| \to 0. \]
Now let \( 0 < \alpha < 1/2 \). Comparing \( T_n(a) \) and \( T_{n+3}(a) \) we see that
\[ T_n(a)P_nV_{n+3}f - \lambda P_nV_{n+3}f = P_nT_{n+3}(a)V_{n+3}f - \lambda P_nV_{n+3}f \]
\[ - \begin{pmatrix} a_{-n} \\ \vdots \\ a_{-1} \end{pmatrix} (V_{n+3}f)_n - \begin{pmatrix} a_{-n-1} \\ \vdots \\ a_{-2} \end{pmatrix} (V_{n+3}f)_{n+1} - \begin{pmatrix} a_{-n-2} \\ \vdots \\ a_{-3} \end{pmatrix} (V_{n+3}f)_{n+2}. \]
As above, the norms of the three columns are seen to be \( o(1) \) in case \( f \) is constant on \([0,\pi]\). If \( f(\pi) = 0 \), then \( (V_{n+3}f)_{n+2} = 0 \),
\[ |(V_{n+3}f)_{n+1}| = \frac{1}{\sqrt{n+2}} \left| f\left(\pi - \frac{\pi}{n+2}\right) \right|, \]
\[ |(V_{n+3}f)_n| = \frac{1}{\sqrt{n+2}} \left| f\left(\pi - \frac{2\pi}{n+2}\right) \right|, \]
which is \( O(1/n^{3/2}) \) whenever \( f \in C^1[0,\pi] \), and hence the norm of the sum of the columns is \( O(\|T_n(a)\|/n^{3/2}) = o(1) \). Thus, in addition to (9) we get
\[ \sqrt{\frac{n+2}{n-1}} \|T_n(a)P_nV_{n+3}f - \lambda P_nV_{n+3}f\|/\|V_nf\| \to 0. \]

We have
\[ \delta_n^2 := \left\| V_nf - 2\sqrt{\frac{n}{n-1}} P_nV_{n+1}f + \sqrt{\frac{n+2}{n-1}} P_nV_{n+3}f \right\|^2 \]
\[ = \frac{2}{n-1} \sum_{j=0}^{n-1} \left| f\left(\frac{j\pi}{n}\right) - 2f\left(\frac{j\pi}{n-1}\right) + f\left(\frac{j\pi}{n+2}\right) \right|^2. \]
If $f \in C^2[0, \pi]$, then
\[
f \left( \frac{j\pi}{n-1} \right) - 2f \left( \frac{j\pi}{n} \right) + f \left( \frac{j\pi}{n+2} \right)
= f' \left( \frac{j\pi}{n} \right) \left( \frac{j\pi}{n-1} - \frac{j\pi}{n} \right) + \frac{f''(\xi)}{2} \left( \frac{j\pi}{n-1} - \frac{j\pi}{n} \right)^2
+ f' \left( \frac{j\pi}{n} \right) \left( \frac{j\pi}{n+2} - \frac{j\pi}{n} \right) + \frac{f''(\eta)}{2} \left( \frac{j\pi}{n+2} - \frac{j\pi}{n} \right)^2
= f' \left( \frac{j\pi}{n} \right) \left( \frac{j\pi(4-n)}{n(n-1)(n+2)} \right) + O \left( \frac{j^2}{n^4} \right)
= O \left( \frac{j}{n^2} \right) + O \left( \frac{j^2}{n^4} \right).
\]

Thus, using that $|a + b|^2 \leq 2(|a|^2 + |b|^2)$ we obtain that
\[
\delta_n^2 \leq \frac{4}{n-1} \sum_{j=0}^{n-1} \left[ O \left( \frac{j^2}{n^4} \right) + O \left( \frac{j^4}{n^8} \right) \right] = O \left( \frac{1}{n^2} \right)
\]
and hence, by (9) and (10),
\[
\frac{\|T_n(a)V_nf - \lambda V_nf\|}{\|V_nf\|} = o(1) + O(\|T_n(a)\|\delta_n) = o(1) + O(n^{2a-1}) = o(1). \quad \Box
\]

4 Exponentials as Pseudomodes

In this section we prove Theorems 1.1 and 1.2(a).

For $f \in C[0, \pi]$, we abbreviate $FV_nf$ to $F_nf$. Thus,
\[
(F_nf)(\theta) = \sum_{\ell=0}^{n-1} (V_nf)_\ell e^{i\ell\theta} \quad (\theta \in \mathbb{R})
\]
and formula (4) amounts to
\[
(T_n(a)V_nf)_j = \frac{1}{2\pi} \int_0^{2\pi} a(\theta)(F_nf)(\theta)e^{-ij\theta} d\theta \quad (j = 0, \ldots, n-1).
\]

We put $e_k(x) = e^{ikx}$.

Lemma 4.1 If $n = 2m + 1$, then
\[
(F_ne_k)(\theta) = -\frac{1}{\sqrt{m}} e^{im(\theta + \pi + \frac{k\pi}{2m})} \frac{\sin \left( m + \frac{1}{2} \right) \left( \theta + \pi + \frac{k\pi}{2m} \right)}{\sin \frac{\theta - \theta_m}{2}}
\]
with $\theta_m = \pi - \frac{k\pi}{2m}$.
Proof. We have
\[
(F_n e_k)(\theta) = \frac{1}{\sqrt{m}} \sum_{\ell=0}^{2m} (-1)^\ell e_k \left( \frac{\ell \pi}{2m} \right) e^{i \theta} = \frac{1}{\sqrt{m}} \sum_{\ell=0}^{2m} e^{i \ell(\pi + \frac{\ell \pi}{2m} + \theta)}
\]

so let
\[
\frac{1}{\sqrt{m}} \left( 1 - e^{i(2m+1)(\theta + \pi + \frac{k \pi}{2m})} \right) = \frac{1}{\sqrt{m}} e^{i m(\theta + \pi + \frac{k \pi}{2m})} \sin \left( m + \frac{1}{2} \right) (\theta + \pi + \frac{k \pi}{2m})
\]
and
\[
\sin \frac{1}{2} \left( \theta + \pi + \frac{k \pi}{2m} \right) = - \sin \frac{1}{2} \left( \theta - \left( \pi - \frac{k \pi}{2m} \right) \right) = - \sin \frac{1}{2} (\theta - \theta_m). \quad \Box
\]

**Theorem 4.2** If $a$ is in $L^1$ and Lipschitz continuous in a neighborhood of $\pi$, the $V_n e_k$ is an asymptotic pseudomode of $T_n(a)$ for $\lambda = a(\pi)$.

**Proof.** Fix $k \in \mathbb{Z}$, let $n = 2m + 1$, suppose $\sqrt{m} \leq j \leq 2m - \sqrt{m}$, and put
\[
h_{m,j}(\theta) = -e^{im(\theta + \pi + \frac{k \pi}{2m})} \sin \left( m + \frac{1}{2} \right) (\theta + \pi + \frac{k \pi}{2m}) e^{-ij\theta}.
\]
Then
\[
h_{m,j}(\theta) = A_m e^{i(2m-j+1/2)\theta} + B_m e^{-i(j+1/2)\theta}
\]
where $A_m$ and $B_m$ have constant modulus $1/2$ and are independent of $j$ and $\theta$. Note that $2m - j + 1/2 \to \infty$ and $j + 1/2 \to \infty$ as $m \to \infty$. By (11) and Lemma 4.1,
\[
\delta_j := (T_n(a)V_n e_k - a(\theta_m)V_n e_k)_j = \frac{1}{2\pi} \int_0^{2\pi} \frac{a(\theta) - a(\theta_m)}{\sin \frac{\theta - \theta_m}{2}} h_{m,j}(\theta) d\theta.
\]
(12)

Put $\varepsilon_m = 1/m^\gamma$ with $0 < \gamma < 1$. If $|\theta - \pi| \leq \varepsilon_m$, then $\left| \frac{\theta - \theta_m}{\varepsilon_m \sin \frac{\theta - \theta_m}{2}} \right| \geq 1/m |\theta - \theta_m|$ for all $m \geq m_1$, and since $|a(\theta) - a(\theta_m)| \leq K |\theta - \theta_m|$, it follows that
\[
\frac{1}{\sqrt{m}} \int_{|\theta - \pi| \leq \varepsilon_m} \left| \frac{a(\theta) - a(\theta_m)}{\sin \frac{\theta - \theta_m}{2}} \right| |h_{m,j}(\theta)| d\theta = O \left( \frac{\varepsilon_m}{m^{3/2}} \right).
\]
(13)

So let $|\theta - \pi| > \varepsilon_m$. In that case $\left| \sin \frac{\theta - \pi}{2} \right| \geq \sin \frac{\varepsilon_m}{2} \geq \frac{\varepsilon_m}{\pi}$ and hence
\[
|q_m| := \frac{-\cos \frac{\theta - \pi}{2} \sin \frac{k \pi}{4m}}{\sin \frac{\theta - \pi}{2} \cos \frac{\theta - \pi}{2}} = O \left( \frac{1}{m \varepsilon_m} \right) = o(1)
\]
uniformly in $\theta$. Thus $|q_m(\theta)| \leq 1/2$ for all $m \geq m_2$. For these $m$'s,
\[
\frac{1}{\sin \frac{\theta - \pi}{2}} = \frac{1}{\sin \frac{\theta - \pi}{2} - \frac{q_m - \pi}{2}} = \frac{1}{\sin \frac{\theta - \pi}{2} + \frac{k \pi}{2m}}
\]
and
\[
\frac{1}{\sin \frac{\theta - \pi}{2} \cos \frac{k \pi}{4m} [1 - q_m(\theta)]} = \frac{1 + p_m(\theta)}{\sin \frac{\theta - \pi}{2} \cos \frac{k \pi}{4m}}
\]

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with
\[ |p_m(\theta)| \leq |q_m(\theta)| \frac{1}{1 - |q_m(\theta)|} = O \left( \frac{1}{m \varepsilon_m} \right) \]
uniformly in \( \theta \). Consequently, for \( |\theta - \pi| > \varepsilon_m \) and \( m \geq m_2 \) we have the decomposition
\[
\frac{a(\theta) - a(\theta_m)}{\sin \frac{\theta - \theta_m}{2}} = \frac{a(\theta) - a(\pi)}{\sin \frac{\theta - \theta_m}{2}} + \frac{a(\theta) - a(\pi)}{\sin \frac{\theta - \theta_m}{2}} p_m(\theta) + \frac{a(\pi) - a(\theta_m)}{\sin \frac{\theta - \theta_m}{2}}.
\]
By the Riemann-Lebesgue theorem,
\[
\int_{0}^{2\pi} \frac{a(\theta) - a(\pi)}{\sin \frac{\theta - \pi}{2}} h_{m,j}(\theta)d\theta \to 0
\]
as \( m \to \infty \) uniformly with respect to \( j \in [\sqrt{m}, 2m - \sqrt{m}] \). Since
\[
\int_{|\theta - \pi| \leq \varepsilon_m} \left| \frac{a(\theta) - a(\pi)}{\sin \frac{\theta - \pi}{2}} \right| d\theta = O(\varepsilon_m) = o(1)
\]
and thus
\[
\int_{|\theta - \pi| \leq \varepsilon_m} \frac{a(\theta) - a(\pi)}{\sin \frac{\theta - \pi}{2}} h_{m,j}(\theta)d\theta \to 0
\]
as \( m \to \infty \) uniformly in \( j \in [\sqrt{m}, 2m - \sqrt{m}] \), we arrive at the conclusion that
\[
\frac{1}{\sqrt{m}} \int_{|\theta - \pi| > \varepsilon_m} \left| \frac{a(\theta) - a(\pi)}{\sin \frac{\theta - \pi}{2}} \right| p_m(\theta) d\theta = O \left( \frac{1}{\varepsilon_m} \right)
\]
(14) as \( m \to \infty \) uniformly in \( j \in [\sqrt{m}, 2m - \sqrt{m}] \). Further,
\[
\frac{1}{\sqrt{m}} \int_{|\theta - \pi| > \varepsilon_m} \left| \frac{a(\theta) - a(\pi)}{\sin \frac{\theta - \pi}{2}} \right| |p_m(\theta)| d\theta 
\leq \frac{1}{\sqrt{m}} O \left( \frac{1}{m \varepsilon_m} \right) \int_{|\theta - \pi| > \varepsilon_m} \left| \frac{a(\theta) - a(\pi)}{\sin \frac{\theta - \pi}{2}} \right| d\theta = O \left( \frac{1}{m^{3/2} \varepsilon_m} \right)
\]
(15)
and
\[
\frac{1}{\sqrt{m}} \int_{|\theta - \pi| > \varepsilon_m} \left| \frac{a(\pi) - a(\theta_m)}{\sin \frac{\theta - \theta_m}{2}} \right| d\theta 
\leq \frac{1}{\sqrt{m}} K |\pi - \theta_m| \int_{|\theta - \pi| > \varepsilon_m} d\theta = O \left( \frac{1}{m^{3/2} \varepsilon_m} \right).
\]
(16)
In summary, from (13) to (16) we obtain that
\[ |\delta_j| \leq \frac{\beta'_m}{m^{1/2}} + C \left( \frac{\varepsilon_m}{m^{1/2}} + \frac{1}{m^{3/2}\varepsilon_m} \right) \]
with \( \beta'_m \to 0 \) and \( C < \infty \) for \( j \in [\sqrt{m}, 2m - \sqrt{m}] \). Choosing \( \gamma = 1/2 \) and thus \( \varepsilon_m = 1/\sqrt{m} \), we get
\[ |\delta_j| \leq \frac{\beta_m}{m^{1/2}} \]
with \( \beta_m \to 0 \) for \( j \in [\sqrt{m}, 2m - \sqrt{m}] \).

To get an estimate for \( j < \sqrt{m} \) and \( j > 2m - \sqrt{m} \), we use (12), which implies that
\[ |\delta_j| \leq \frac{1}{2\pi\sqrt{m}} \int_0^{2\pi} \frac{|a(\theta) - a(\theta_m)|}{|\sin \frac{\theta - \theta_m}{2}|} d\theta. \]
The function inside the integral is bounded in a neighborhood of \( \theta_m \) and bounded by a constant times \( |a(\theta) - a(\theta_m)| \) outside this neighborhood. Hence \( |\delta_j| = O(1/\sqrt{m}) \) for all \( j \) and in particular, for \( j < \sqrt{m} \) and \( j > 2m - \sqrt{m} \).

Putting things together we obtain that
\[ ||\delta||^2 = \sum_{j=0}^{2m} |\delta_j|^2 = \sum_{j<\sqrt{m}} |\delta_j|^2 + \sum_{\sqrt{m} \leq j \leq 2m-\sqrt{m}} |\delta_j|^2 + \sum_{2m-\sqrt{m} \leq j} |\delta_j|^2 \]
\[ = \sqrt{m} O \left( \frac{1}{m} \right) + (2m - 2\sqrt{m}) \frac{\beta_m^2}{m} + \sqrt{m} O \left( \frac{1}{m} \right) = o(1). \]
As
\[ ||T_n(a)V_ne_k - a(\pi)V_ne_k|| \leq ||\delta|| + |a(\pi) - a(\theta_m)| \||V_ne_k|| \]
with \( |a(\pi) - a(\theta_m)| \to 0 \) and
\[ ||V_ne_k||^2 = \frac{2}{n-1} \sum_{j=0}^{n-1} \left| e_k \left( \frac{j\pi}{n-1} \right) \right|^2 = 2, \]
it follows that \( ||T_n(a)V_ne_k - a(\pi)V_ne_k||/||V_ne_k|| \to 0 \) as \( n = 2m + 1 \to \infty \). Lemma 3.3 completes the proof. \( \square \)

**Proof of Theorem 1.1.** Combining Theorem 4.2 and Proposition 3.2 we arrive at the conclusion that \( \tilde{V}_n e_k \) is an asymptotic pseudomode of \( T_n(a) \) for \( \lambda = a(\pi) \). Thus, if \( f = \sum_{k=-m}^{m} c_ke_k \) is an arbitrary nonzero trigonometric polynomial, then
\[ ||T_n(a)\tilde{V}_n f - a(\pi)\tilde{V}_n f|| \leq \sum_{k=-m}^{m} |c_k| \||T_n(a)\tilde{V}_n e_k - a(\pi)\tilde{V}_n e_k|| = o(1). \]
This together with (5) shows that \( \tilde{V}_n f \) is an asymptotic pseudomode of \( T_n(a) \) for \( \lambda = a(\pi) \). \( \square \)
Proof of Theorem 1.2(a). Given $\delta > 0$, there is a trigonometric polynomial $p$ such that the maximum of $|f(x) - p(x)|$ on $[0, \pi]$ is smaller than $\delta$. By the triangle inequality,

$$\frac{\|T_n(a)\vec{V}_n f - a(\pi)\vec{V}_n f\|}{\|\vec{V}_n f\|} \leq \frac{\|T_n(a)\vec{V}_n p - a(\pi)\vec{V}_n p\|}{\|\vec{V}_n f\|} + \left(\|T_n(a)\| + |a(\pi)|\right)\frac{\|\vec{V}_n (f - p)\|}{\|\vec{V}_n f\|}. \tag{17}$$

Using (5) and taking into account that $\vec{V}_n p$ is an asymptotic pseudomode by Theorem 1.1, we see that the first term on the right of (17) is smaller than $\varepsilon/2$ if only $\delta < \delta_1$ and $n \geq n_1$. Since $\|T_n(a)\| \leq \|a\|_\infty$, it follows from (5) that the second term on the right of (17) is smaller than $\varepsilon/2$ if $\delta < \delta_2$ and $n \geq n_2$. Thus, for $n \geq \max(n_1, n_2)$ the left-hand side of (17) does not exceed an arbitrarily prescribed $\varepsilon > 0$. \hfill $\square$

5 Sines as Pseudomodes

We now turn to the proof of Theorem 1.2(b). We put $\varphi_k(x) := \sin kx$. Throughout this section $k \geq 1$ is a natural number.

Lemma 5.1 Let $n = 2m + 1$. Then

$$(F_n \varphi_k)(\theta) = \frac{1}{\sqrt{m}} \frac{e^{i\theta} \sin \frac{k\pi}{2m}}{\cos \theta + \cos \frac{\pi}{2m}}$$

if $k$ is odd and

$$(F_n \varphi_k)(\theta) = -\frac{i}{\sqrt{m}} \frac{e^{i\theta} \sin \frac{k\pi}{2m}}{\cos \theta + \cos \frac{\pi}{2m}}$$

if $k$ is even.

Proof. We have to compute

$$(F_n \varphi_k)(\theta) = \frac{1}{\sqrt{m}} \sum_{j=0}^{2m} (-1)^j \sin \frac{jk\pi}{2m} e^{ij\theta}.$$ 

This can be done straightforwardly by summing up products of sines and cosines in the usual way. Alternatively, one can employ the formula

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\cos N\theta}{\cos \theta - \cos a} d\theta = \frac{\sin |N|a}{\sin a}$$
(see, e.g., [13, 2.5.16.23]). Let, for example, \( k \) be odd. Then the formula gives

\[
\frac{1}{2\pi} \int_{0}^{2\pi} e^{im\theta} \cos m\theta e^{-ij\theta} d\theta
= \frac{1}{2\pi} \int_{0}^{2\pi} \cos m\theta \cos(m-j)\theta d\theta
= \frac{1}{4\pi} \int_{0}^{2\pi} \cos j\theta + \cos(2m-j)\theta d\theta
= \frac{1}{\sin \left(\pi - \frac{k\pi}{2m}\right)} \left[ \sin |j| \left(\pi - \frac{k\pi}{2m}\right) + \sin |2m-j| \left(\pi - \frac{k\pi}{2m}\right) \right].
\]

We have \( \sin(\pi - \frac{k\pi}{2m}) = \sin \frac{k\pi}{2m} \), and the term in brackets is

\[-2(-1)^j \sin \frac{jk\pi}{2m}\]

for \( 0 \leq j \leq 2m \) and zero for \( j < 0 \) or \( j > 2m \).

Theorem 1.1 implies that, under the hypotheses of the following lemma, \( V_n \phi_k \) is an asymptotic pseudomode of \( T_n(a) \) for \( \lambda = a(\pi) \). However, in order to prove Theorem 1.2(b) we need a little more.

**Lemma 5.2** Let \( a \in L^1 \), suppose \( a \) is Lipschitz continuous in a neighborhood of \( \pi \), and assume \( a(\pi - \theta) = a(\pi + \theta) \) for sufficiently small \( \theta \). Let also \( \|T_n(a)\| = O(n^{2\alpha}) \) for some \( \alpha \in [0, 1/2] \). If \( 0 < \beta < 1 \) and \( k \leq m^\beta \), then

\[
\|T_n(a)V_{2m+1}\phi_k - a(\pi)V_{2m+1}\phi_k\| \leq \frac{C}{m^{(1-\beta)/3}}
\]

with some constant \( C < \infty \) depending only on \( a \) and \( \beta \).

**Proof.** Let \( n = 2m+1 \). We again abbreviate \( \pi - \frac{k\pi}{2m} \) to \( \theta_m \). By (11),

\[
\delta_j := [T_n(a)V_n \phi_k - a(\theta_m)V_n \phi_k]_j = \frac{1}{2\pi} \int_{0}^{2\pi} [a(\theta) - a(\theta_m)](F_n \phi_k)(\theta)e^{-ij\theta} d\theta.
\]

From Lemma 5.1 we infer that

\[
|\delta_j| \leq \frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{a(\theta) - a(\theta_m)}{\cos \theta - \cos \theta_m} \right| d\theta \frac{1}{\sqrt{m}} \sin \frac{k\pi}{2m}
\]

(19)

Let \( \gamma = (1 - \beta)/3 \) and put \( \varepsilon_m = 1/m^\gamma \). Suppose \( k \leq m^\beta \). If \( |\theta - \pi| \geq \varepsilon_m \), then

\[
\cos \theta - \cos \theta_m \geq \cos(\pi - \varepsilon_m) - \cos \theta_m = \cos(\pi - \varepsilon_m) - \cos \left(\pi - \frac{k\pi}{2m}\right)
= 2 \sin \left(\frac{\varepsilon_m}{2} + \frac{k\pi}{2m}\right) \sin \left(\frac{\varepsilon_m}{2} - \frac{k\pi}{2m}\right) \geq C_1 \varepsilon_m^2
\]

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for all sufficiently large $m$, because $\beta + \gamma < 1$ and hence $k/(m\varepsilon_m) = O(m^{\beta+\gamma-1}) = o(1)$. Here and in the following $C_i$ ($i = 1, 2, \ldots$) denotes a finite constant that depends only on $a$ and $\beta$. Thus,

$$
\int_{|\theta - \pi| \geq \varepsilon_m} \left| \frac{a(\theta) - a(\theta_m)}{\cos \theta - \cos \theta_m} \right| d\theta \leq \frac{C_2}{\varepsilon_m^2}. \quad (20)
$$

Let $|a(t) - a(s)| \leq K|t - s|$ for $t, s \in [\pi - \mu, \pi + \mu]$ and $a(\pi - \theta) = a(\pi + \theta)$ for $|\theta| \leq \mu$. If $m$ is large enough, then $\varepsilon_m < \mu$ and hence

$$
\int_{|\theta - \pi| < \varepsilon_m} \left| \frac{a(\theta) - a(\theta_m)}{\cos \theta - \cos \theta_m} \right| d\theta = 2 \int_{|\theta - \pi| < \varepsilon_m} \left| \frac{a(\theta) - a(\theta_m)}{\cos \theta - \cos \theta_m} \right| d\theta \\
\leq 2K \int_{|\theta - \pi| < \varepsilon_m} \left| \frac{\theta - \varepsilon_m}{\cos \theta - \cos \theta_m} \right| d\theta = 2K \int_{|\theta - \pi| < \varepsilon_m} \left| \frac{\theta - \varepsilon_m}{\sin \frac{\theta - \varepsilon_m}{2}} \right| d\theta \\
\leq K \pi \int_{|\theta - \pi| < \varepsilon_m} \frac{d\theta}{\sin \frac{\theta - \varepsilon_m}{2}} \leq K \pi \int_{|\theta - \pi| < \varepsilon_m} \frac{d\theta}{\sin \frac{m\varepsilon_m}{4m}} \\
= \frac{K \pi \varepsilon_m}{\sin \frac{m\varepsilon_m}{4m}} \leq C_3 \frac{m\varepsilon_m}{k} \quad (21)
$$

Inserting (20) and (21) in (19) we get

$$
|\delta_j| \leq C_4 \left( \frac{1}{\varepsilon_m^2} + \frac{m\varepsilon_m}{k} \right) \frac{k}{m^{3/2}} = C_4 \left( \frac{k}{m^{3/2} \varepsilon_m^2} + \varepsilon_m \right) \frac{1}{m^{3/2-\beta-\gamma} + \frac{1}{m^{\gamma+1/2}}}.
$$

Since $3/2 - \beta - 2\gamma = \gamma + 1/2 = 5/6 - \beta/3$, we arrive at the conclusion that $|\delta_j| = O(1/m^{5/6-\beta/3})$. It follows that

$$
\|T_n(a)V_n \varphi_k - a(\theta_m) V_n \varphi_k\| = \left( \sum |\delta_j|^2 \right)^{1/2} = O \left( m \frac{1}{m^{10/6-2\beta/3}} \right)^{1/2} = O \left( \frac{1}{m^{(1-\beta)/3}} \right).
$$

Since

$$
\|a(\pi) - a(\theta_m)\| V_n \varphi_k \| \leq K|\pi - \theta_m| \|V_n \varphi_k\| \leq \frac{C_5}{m^{1-\beta}} \|V_n \varphi_k\|
$$

and, by (5), $\|V_n \varphi_k\| \to 1$, we obtain (18). \qed

**Lemma 5.3** Let $a \in L^1$, let $a$ be Lipschitz continuous in a neighborhood of $\pi$, and let $a(\pi - \theta) = a(\pi + \theta)$ for all sufficiently small $\theta$. Suppose $\|T_n(a)\| = O(n^{2\alpha})$ for some $\alpha < 1/2$. If $f$ is the restriction to $[0, \pi]$ of a $2\pi$-periodic and odd function in $C^3(\mathbb{R})$, then $V_n f$ is an asymptotic pseudomode of $T_n(a)$ for $\lambda = a(\pi)$.  

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Proof. Fix $\beta \in (0, 1)$. By Lemma 3.3 we may assume that $n = 2m + 1$. Suppose $f_n$ is a trigonometric polynomial of the form

$$f_n(x) = \sum_{k=1}^{m^3} c_k \sin kx. \quad (22)$$

The vectors $V_n \varphi_1, \ldots, V_n \varphi_n$ form an orthogonal basis of $\mathbb{C}^n$. Hence

$$\|V_n f_n\|^2 = \sum_{k=1}^{m^3} |c_k|^2 = \frac{2}{\pi} \int_0^\pi |f_n(x)|^2\,dx =: \gamma_n^2.$$  

From Lemma 5.2 we therefore get

$$\|T_n(a)V_n f_n - a(\pi)V_n f_n\|/\|V_n f_n\| \leq \frac{1}{\gamma_n} \sum_{k=1}^{m^3} |c_k| \|T_n(a)V_n \varphi_k - a(\pi)V_n \varphi_k\| \leq \frac{C_1}{\gamma_n} \frac{1}{n^{(1-\beta)/3}} \sum_{k=1}^{m^3} |c_k| \frac{1}{\gamma_n} \frac{1}{n^{(1-\beta)/3}} \frac{m^3}{2} \left( \sum_{k=1}^{m^3} |c_k|^2 \right)^{1/2} \leq C_2 \frac{1}{n^{(1-\beta)/3-\beta/2}} = C_2 \frac{1}{n^{1/3-5\beta/6}}. \quad (23)$$

Now consider the function $f$ and let

$$E_N(f) = \inf \{ \|f - c_1 \varphi_1 - \ldots - c_N \varphi_N\|_\infty : c_1, \ldots, c_N \in \mathbb{C} \},$$

where $\|g\|_\infty$ is the maximum of $|g(x)|$ on $[0, \pi]$. We can find $f_n$ of the form (22) such that

$$\|f - f_n\|_\infty \leq 2 E_m(f).$$

Clearly,

$$\|T_n(a)V_n f - a(\pi)V_n f\|/\|V_n f\| \leq \|T_n(a)V_n f_n - a(\pi)V_n f_n\|/\|V_n f\| + \|T_n(a)V_n (f - f_n) - a(\pi)V_n (f - f_n)\|/\|V_n f\|.$$  

Let us denote the sum on the right by $A_1 + A_2$. We have

$$\|V_n f\|^2 = \frac{2}{\pi} \int_0^\pi |f(x)|^2\,dx + o(1),$$

$$\|V_n f_n\|^2 = \frac{2}{\pi} \int_0^\pi |f_n(x)|^2\,dx = \frac{2}{\pi} \int_0^\pi |f(x)|^2\,dx + o(1).$$

Thus, by (23),

$$A_1 \leq \frac{C_2}{n^{1/3-5\beta/6}} \frac{\|V_n f_n\|}{\|V_n f\|} = O \left( \frac{1}{n^{1/3-5\beta/6}} \right).$$
On the other hand,
\[ \|V_n(f - f_n)\|^2 = \frac{1}{m} \sum_{j=0}^{2m} \left| f\left(\frac{j\pi}{2m}\right) - f_n\left(\frac{j\pi}{2m}\right) \right|^2 \leq 2\|f - f_n\|_\infty^2 \leq 8E_{m^\beta}(f) \]
and hence
\[ A_2 \leq (\|T_n(a)\| + |a(\pi)|)\|V_n(f - f_n)\|/\|\|V_n\| = O\left(n^{2\alpha}E_{m^\beta}(f)\right). \]

Suppose the odd and 2\pi-periodic extension of \( f \) is in \( C^k(R) \). Then \( E_{m^\beta}(f) = O\left(\frac{1}{m^\beta k}\right) \). Since \( O\left(\frac{1}{n^{1/3 - 5\beta/6}}\right) = o(1) \) for \( \beta < \frac{2}{5} \) and \( O(n^{2\alpha}E_{m^\beta}(f)) = O\left(\frac{1}{m^{\beta k - 2\alpha}}\right) = o(1) \) for \( \beta > \frac{2\alpha}{k} \), we arrive at the conclusion that \( A_1 + A_2 = o(1) \) whenever \( 2\alpha/k < 2/5 \), that is, \( k > 5\alpha \). This is certainly satisfied for \( k = 3 \). □

**Proof of Theorem 1.1(b).** By Proposition 3.2, it is sufficient to prove the theorem for \( V_n f \) instead of \( \tilde{V}_n f \).

Let \( f \in C^3[0, \pi] \). A function in \( C^3[0, \pi] \) can be extended to an odd and 2\pi-periodic function in \( C^3(R) \) if and only if its values and second derivatives at 0 and \( \pi \) are zero. Thus, let

\[ g(x) = f(x) - A - B \cos x - C \cos 2x - D \cos 3x. \]

Then
\[
\begin{align*}
g(0) &= f(0) - A - B - C - D, & g(\pi) &= f(\pi) - A + B - C + D, \\g''(0) &= f''(0) + B + 4C + 9D, & g''(\pi) &= f''(\pi) - B + 4C - 9D, \end{align*}
\]
and since
\[
\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & -1 & -4 & -9 \\ 0 & 1 & -4 & 9 \end{pmatrix} = 128 \neq 0,
\]
there are \( A, B, C, D \) such that \( g(0) = g(\pi) = g''(0) = g''(\pi) = 0 \). Put \( \psi_k(x) = \cos kx \).

Lemma 5.3 and Theorem 4.2 imply that \( V_n g, V_n \psi_0, V_n \psi_1, V_n \psi_2, V_n \psi_3 \) are asymptotic pseudomodes. Hence \( V_n f \) is also an asymptotic pseudomode. □

Using translation invariance, we can now easily construct asymptotic pseudomodes at all “regular” values of the generating function of the Toeplitz matrices.

**Theorem 5.4** Let \( a \in L^1 \), let \( \|T_n(a)\| = O(n^{2\alpha}) \) for some \( \alpha \in [0, 1/2) \), and suppose the 2\pi-periodic extension of \( a \) is Lipschitz continuous in a neighborhood of \( \theta_0 \in R \).
Assume also that at least one of the following holds: (a) $f$ is a trigonometric polynomial, (b) $a \in L^\infty$ and $f \in C[0, \pi]$, (c) $a(\pi - \theta) = a(\pi + \theta)$ for all sufficiently small $\theta$, and $f \in C^3[0, \pi]$. Then the sequences $\{\tilde{V}_{n, \theta_0}f\}$ and $\{V_{n, \theta_0}f\}$ given by

$$
\tilde{V}_{n, \theta_0}f = \left( \frac{e^{-ij\theta_0}}{\sqrt{n+1}} f \left( \frac{j\pi}{n+1} \right) \right)^n_{j=1}, \quad V_{n, \theta_0}f = \left( \frac{e^{-ij\theta_0}}{\sqrt{n-1}} f \left( \frac{j\pi}{n-1} \right) \right)_{j=0}^{n-1}
$$

are both asymptotic pseudomodes of $\{T_n(a)\}$ for $\lambda = a(\theta_0)$.

**Proof.** We have $V_{n, \theta_0}f = D_{\theta_0}V_nf$ where $D_{\theta_0}$ is the unitary diagonal matrix

$$
D_{\theta_0} = \text{diag} (1, e^{i(\pi-\theta_0)}, \ldots, e^{i(n-1)(\pi-\theta_0)}).
$$

Consequently,

$$
\|T_n(a)V_{n, \theta_0}f - a(\theta_0)V_{n, \theta_0}f\| = \|T_n(a)D_{\theta_0}V_nf - a(\theta_0)D_{\theta_0}V_nf\| = \|D_{\theta_0}^{-1}T_n(a)D_{\theta_0}V_nf - a(\theta_0)V_nf\|. \quad (24)
$$

It can be readily verified that $D_{\theta_0}^{-1}T_n(a)D_{\theta_0} = T_n(a_{\theta_0})$ where $a_{\theta_0}(\theta) = a(\theta + \theta_0 - \pi)$. Since $a_{\theta_0}(\pi) = a(\theta_0)$, we see that (24) is $\|T_n(a_{\theta_0})V_nf - a_{\theta_0}(\pi)V_nf\|$. Theorems 1.1 and 1.2 in conjunction with Proposition 3.2 therefore imply that $V_{n, \theta_0}f$ and $\tilde{V}_{n, \theta_0}f$ are asymptotic pseudomodes for $a(\theta_0)$.  □

6 Pseudomodes from inside the Fourier basis

In this section we prove Theorem 1.3. Let $\theta_0 = 2\pi(1 - \beta)$, $e_{2k}(x) = e^{2\pi ikx}$, and $n = \beta n + k_n$ with $k_n \in [-k_0, k_0]$. Define $a_{\theta_0}$ by $a_{\theta_0}(\theta) = a(\theta + \theta_0 - \pi)$ as in the proof of Theorem 5.4. Since $\|V_ne_{2kn}\| = \sqrt{2}$, Theorem 4.2 gives

$$
\|T_n(a_{\theta_0})V_ne_{2kn} - a_{\theta_0}(\pi)V_ne_{2kn}\| \to 0. \quad (25)
$$

For $f \in C[0, \pi]$, put

$$
V_0^nf = \left( \frac{(-1)^j}{\sqrt{n}} f \left( \frac{j\pi}{n} \right) \right)_{j=0}^{n-1}.
$$

The argument of the proof of Proposition 3.2 shows that in (25) we may replace $V_n$ by $V_0n$. Hence, with $D_{\theta_0}$ as in the proof of Theorem 5.4,

$$
\|T_n(a)D_{\theta_0}V_0ne_{2kn} - a(\theta_0)D_{\theta_0}V_0ne_{2kn}\| \to 0. \quad (26)
$$

Since

$$
(D_{\theta_0}V_0ne_{2kn})_j = e^{ij(\pi-\theta_0)} \left( \frac{(-1)^j}{\sqrt{n}} \right) e^{2\pi ikj/n} = \frac{1}{\sqrt{n}} e^{-ij\theta_0} e^{2\pi \frac{n}{n+2k} j}
$$

$$
= \frac{1}{\sqrt{n}} e^{2\pi \frac{n}{n+2k} (\frac{n}{n+2k} + k)} j = \frac{1}{\sqrt{n}} e^{2\pi \frac{n}{n+2k} (\beta n + k)} j = (U_n e_{\beta n + k} j),
$$

we obtain from (26) that $\|T_n(a)U_{n}e_{\beta n + k_n} - a(\theta_0)U_{n}e_{\beta n + k_n}\| \to 0$, which is just the assertion.
References


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