Automates cellulaires probabilistes
et
mesures spécifiques sur des espaces symboliques

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Thèse effectuée sous la direction de Jean Mairesse

Vendredi 22 novembre 2013
Example of PCA

- with probability \(\frac{1}{2}\)
- with probability \(\frac{1}{2}\)
- (with probability 1)

or

or
Example of PCA

- With probability 1/2
- With probability 1/2

(with probability 1)

or

or
Example of PCA

- with probability 1/2
- with probability 1/2
- (with probability 1)

or

or

□ □ □ □ ■ □ □ □ □ ■ □ □ □ ■ □ □ □ ■ □ □ □
□ ■ □ ■ □ □ □ ■ ■ □ □ ■ ■ □ □ ■ □ □ □ □ ■

 Irène Marcovici

PCA and specific measures on symbolic spaces
Example of PCA
Example of PCA

\[\begin{array}{c}
\text{with probability } \frac{1}{2} \\
\text{with probability } \frac{1}{2}
\end{array}\]

\[\begin{array}{c}
\text{(with probability } 1) \\
\text{or } 0.5 \\
\text{or } 0.5
\end{array}\]
Example of PCA

PCA of set of cells $E = \mathbb{Z}$,
Example of PCA

PCA of set of cells $E = \mathbb{Z}$, alphabet $\mathcal{A} = \{\blacksquare, \square\}$.
Example of PCA

PCA of set of cells $E = \mathbb{Z}$, alphabet $A = \{\blacklozenge, \square\}$, neighbourhood $\mathcal{N} = \{0, 1\}$.
Example of PCA

PCA of set of cells $E = \mathbb{Z}$, alphabet $A = \{■, □\}$, neighbourhood $\mathcal{N} = \{0, 1\}$.

Local function:

$$f : \{■, □\}^2 \to \mathcal{M}(\{■, □\})$$

defined by:

$$f(□□) = \frac{1}{2} \delta □ + \frac{1}{2} \delta ■$$

$$f(□■) = f(■□) = f(■■) = \delta □$$

Ergodicity and perfect sampling
PCA having a specific behaviour
Measures on subshift of finite type
Example of PCA

PCA of set of cells $E = \mathbb{Z}$, alphabet $A = \{\bullet, \square\}$, neighbourhood $\mathcal{N} = \{0, 1\}$.

Local function:

$$f : \{\bullet, \square\}^2 \rightarrow \mathcal{M}(\{\bullet, \square\})$$

Global function:

$$F : \mathcal{M}(\{\bullet, \square\}^{\mathbb{Z}}) \rightarrow \mathcal{M}(\{\bullet, \square\}^{\mathbb{Z}})$$

$$\mu \mapsto \mu F$$
Definition of PCA

Let $\mathcal{A}$ be a finite set called the alphabet.

A PCA $F$ of set of cells $E = \mathbb{Z}^d$ is defined by

- a finite neighbourhood $\mathcal{N} \subset E$,
- a local function $f : \mathcal{A}^{\mathcal{N}} \rightarrow \mathcal{M}(\mathcal{A})$.

From the configuration $(x_k)_{k \in E} \in \mathcal{A}^E$, cell $k$ is updated by the symbol $y$ with probability:

$$f((x_k+v)_{v \in \mathcal{N}})(y),$$

simultaneously and independently of the other cells.
Motivations for the study of PCA.

\[ \text{PCA} = \begin{cases} 
\text{synchronous analogous of interacting particle systems}, \\
\text{natural extension of deterministic CA.}
\end{cases} \]
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PCA = \left\{ \begin{array}{l}
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- **Computer science:** what can we compute in the presence of random errors?
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- Modelling tool in **physics** and in the **life sciences**.
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- **Computer science**: what can we compute in the presence of random errors?
- Modelling tool in **physics** and in the **life sciences**.
- Link with different problems in **probability, combinatorics, symbolic dynamics**.
Back to the example
Back to the example
Notion of ergodicity

The system **forgets** its initial configuration. We say it is **ergodic**.
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**Ergodicity**

A PCA $F$ on $\mathcal{A}^E$ is **ergodic** if:

- it has a **unique invariant measure** $\pi \in \mathcal{M}(\mathcal{A}^E)$, such that $\pi F = \pi$,
- for any initial measure $\mu \in \mathcal{M}(\mathcal{A}^E)$, the sequence of iterates $(\mu F^n)_{n \geq 0}$ **converges** weakly to $\pi$. 

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PCA and specific measures on symbolic spaces
Example of PCA

Here, we can describe explicitly the unique invariant measure $\pi$ of the PCA.
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\[
\pi_{\square} = \frac{\varphi^2}{1 + \varphi^2} \quad \text{and} \quad \pi_{\blacksquare} = \frac{1}{1 + \varphi^2} \quad \left(\varphi = \frac{1 + \sqrt{5}}{2}\right).
\]
Example of PCA

Here, we can describe explicitly the unique invariant measure $\pi$ of the PCA. It is the Markov measure given by:

\[
\begin{align*}
\pi_{\square} &= \frac{\sqrt{5} - 1}{\sqrt{5} + 1}, \\
\pi_{\blacksquare} &= \frac{1}{\sqrt{5} + 1}
\end{align*}
\]

with $\varphi = \frac{1 + \sqrt{5}}{2}$.

But for general PCA, the ergodicity is difficult to determine, and we have no expression of the invariant measure(s)!
1. The ergodicity of CA and hence of PCA is **undecidable**.
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Plan

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2. Inverse problems:
   - PCA having Bernoulli (or Markov) invariant measures,
   - PCA classifying the density.
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3. Measures of maximal entropy of subshift of finite type (SFT). They are also invariant measures of a well-suited PCA.
Plan

1. Ergodicity and perfect sampling
   - Undecidability of the ergodicity
   - Perfect sampling

2. PCA having a specific behaviour
   - Bernoulli invariant measures
   - Density classification

3. Measures on subshift of finite type
   - One-dimensional SFT and the Parry measure
   - Link with PCA
Ergodicity of deterministic CA

A deterministic CA $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is nilpotent if there exists $\alpha \in \mathcal{A}$ such that: $\forall x \in \mathcal{A}^{\mathbb{Z}^d}, \exists n \in \mathbb{N}, F^n(x) = \alpha^{\mathbb{Z}^d}$.
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For deterministic CA, ergodicity $\iff$ nilpotency.

Proof. $\Leftarrow$ is easy; $\Rightarrow$ in two steps:

1. the unique invariant measure has to be a measure concentrated on a monochromatic configuration $\alpha^{\mathbb{Z}^d}$,
2. the convergence properties then implies the nilpotency (using [Guillon & Richard 2008], and [Salo 2012] for $d \geq 2$).
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**Corollary (with [Kari 1992])**

The ergodicity of one-dimensional deterministic CA (and hence of PCA) is undecidable.
Perfect sampling for PCA

Let $F$ be an ergodic PCA of invariant measure $\pi$. In general, we have no explicit description of $\pi$.

**Perfect sampling** of $\pi$: probabilistic algorithm returning a sequence $a_1 \ldots a_n$ with *exactly* the probability it has to appear under the measure $\pi$. 
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**Aim**: simulating the behaviour of the PCA after an infinity of iterations with a (hopefully) finite-time algorithm.
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**Aim:** simulating the behaviour of the PCA after an infinity of iterations with a (hopefully) finite-time algorithm.

**Idea:** adapt the coupling from the past algorithm [Propp-Wilson 1996], with the introduction of a bounding process called the envelope PCA.
Update function of a PCA

A way to run a PCA (on $\mathcal{A} = \{0, 1\}$) from configuration $x \in \mathcal{A}^\mathbb{Z}$:

- generate for each cell $k$ independently and uniformly a random number $r_k$ in $[0, 1]$,
- choose the new state of the cell $k$ to be
  \begin{align*}
  0 & \text{ if } r_k < f((x_{k+v})_{v \in \mathbb{N}})(0), \text{ and } 1 \text{ otherwise.}
  \end{align*}
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Update function of a PCA

A way to run a PCA (on $A = \{0, 1\}$) from configuration $x \in A^\mathbb{Z}$:

- generate for each cell $k$ independently and uniformly a random number $r_k$ in $[0, 1]$,
- choose the new state of the cell $k$ to be
  - 0 if $r_k < f((x_{k+v})_{v \in \mathbb{N}})(0)$, and
  - 1 otherwise.

It defines an update function for $F$, given by:

$$\phi : A^\mathbb{Z} \times [0, 1]^\mathbb{Z} \to A^\mathbb{Z}$$

$$\phi(x, r)_k = \begin{cases} 0 & \text{if } r_k < f((x_i)_{i \in k+\mathbb{N}})(0) \\ 1 & \text{otherwise.} \end{cases}$$
Update function of a PCA

Example: $\mathcal{A} = \{0, 1\}$, neighbourhood $\mathcal{N} = \{0, 1\}$
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Update function of a PCA

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Example: $\mathcal{A} = \{0, 1\}$, neighbourhood $\mathcal{N} = \{0, 1\}$
Envelope PCA

Introduction of an envelope PCA defined on the alphabet

\[ \mathcal{B} = \{0 = \{0\}, 1 = \{1\}, ? = \{0, 1\}\}, \]

to handle configurations partially known.

The update function \( \tilde{\phi} \) of \( \text{env}(P) \) satisfies for \( x \in \mathcal{A}^E \) and \( y \in \mathcal{B}^E \),

\[ x \in y \implies \forall r \in [0, 1]^E, \phi(x, r) \in \tilde{\phi}(y, r). \]
Coupling from the past algorithm

Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$. 
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Let \( F \) be an ergodic PCA on \( E = \mathbb{Z} \), \( \mathcal{A} = \{0, 1\} \), with \( \mathcal{N} = \{0, 1\} \).
Coupling from the past algorithm

Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

$\frac{1}{2} < \frac{1}{2}$

$(r^{1}_i)_{0 \leq i \leq 2}$
Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $A = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

\[(r_i^1)_{0 \leq i \leq 2}\]
Coupling from the past algorithm

Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $A = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

\begin{align*}
? & 1 \\
? & ? & ? \\
? & ? & ? & ?
\end{align*}

$$(r^1_i)_{0 \leq i \leq 2}$$

$$(r^2_i)_{0 \leq i \leq 3}$$
Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

\[ (r_i^1)_{0 \leq i \leq 2} \]
\[ (r_i^2)_{0 \leq i \leq 3} \]
Coupling from the past algorithm

Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

$\begin{align*}
? & 1 \\
? & 0 \ 1 \\
? & ? \ ? \ ? \ ? \\
\end{align*}$

$(r_i^1)_{0 \leq i \leq 3}$

$(r_i^2)_{0 \leq i \leq 3}$
Coupling from the past algorithm

Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $A = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$. 

\[
\begin{align*}
? & \quad 1 \\
? & \quad 0 \quad 1 \\
? & \quad ? \quad ? \quad ? \\
? & \quad ? \quad ? \quad ? \quad ? \\
? & \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ?
\end{align*}
\]

$(r^1_i)_{0 \leq i \leq 2}$

$(r^2_i)_{0 \leq i \leq 3}$

$(r^3_i)_{0 \leq i \leq 4}$
Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

\[
\begin{array}{c}
\text{? 1} \\
\text{? 0 1} \\
\text{1 ? ? 0} \\
\text{? ? ? ? ?}
\end{array}
\begin{array}{c}
(r^1_i)_{0 \leq i \leq 2} \\
(r^2_i)_{0 \leq i \leq 3} \\
(r^3_i)_{0 \leq i \leq 4}
\end{array}
\]
Coupling from the past algorithm

Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

\[
\begin{array}{c}
? & 1 \\
? & 0 & 1 \\
1 & ? & ? & 0 \\
\end{array}
\]

\[
\begin{align*}
(r_1^1)_{0 \leq i \leq 2} \\
(r_2^2)_{0 \leq i \leq 3} \\
(r_3^3)_{0 \leq i \leq 4}
\end{align*}
\]
Coupling from the past algorithm

Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $A = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

$$
\begin{array}{c}
? ? 1 \\
? 0 \ 1 \\
1 ? ? \ 0 \\
\end{array}
\begin{array}{c}
(r_1^1)_{0 \leq i \leq 2} \\
(r_1^2)_{0 \leq i \leq 3} \\
(r_1^3)_{0 \leq i \leq 4} \\
\end{array}
$$
Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $A = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$. 

\[
\begin{align*}
? & \quad 1 \\
? & \quad 0 \quad 1 \\
1 & \quad ? \quad ? \quad 0 \\
? & \quad ? \quad ? \quad ? \\
? & \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \\
\end{align*}
\]

$(r^1_i)_{0 \leq i \leq 2}$, $(r^2_i)_{0 \leq i \leq 3}$, $(r^3_i)_{0 \leq i \leq 4}$, $(r^4_i)_{0 \leq i \leq 5}$
Coupling from the past algorithm

Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $A = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

\[
\begin{array}{cccccc}
? & 1 \\
? & 0 & 1 \\
1 & ? & ? & 0 \\
? & 0 & ? & 1 & ? \\
\end{array}
\]

$(r_i^1)_{0 \leq i \leq 2}$

$(r_i^2)_{0 \leq i \leq 3}$

$(r_i^3)_{0 \leq i \leq 4}$

$(r_i^4)_{0 \leq i \leq 5}$
Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

\[
\begin{align*}
? & \quad 1 \\
? & \quad 0 \quad 1 \\
1 & \quad 1 \quad ? \quad 0 \\
? & \quad 0 \quad ? \quad 1 \quad ? \\
? & \quad ? \quad ? \quad ? \quad ?
\end{align*}
\]

\[(r_i^1)_{0 \leq i \leq 2} \quad (r_i^2)_{0 \leq i \leq 3} \quad (r_i^3)_{0 \leq i \leq 4} \quad (r_i^4)_{0 \leq i \leq 5}\]
Coupling from the past algorithm

Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

\[
\begin{array}{cccccccc}
? & 0 & 1 \\
1 & 0 & 1 & (r^1_i)_{0 \leq i \leq 2} \\
1 & 1 & ? & 0 & (r^2_i)_{0 \leq i \leq 3} \\
? & 0 & ? & 1 & ? & (r^3_i)_{0 \leq i \leq 4} \\
? & ? & ? & ? & ? & (r^4_i)_{0 \leq i \leq 5}
\end{array}
\]
Let $F$ be an ergodic PCA on $E = \mathbb{Z}$, $A = \{0, 1\}$, with $N = \{0, 1\}$.
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$\begin{array}{c}
0 & 1 \\
1 & 0 & 1 \\
1 & 1 & ? & 0 \\
? & 0 & ? & 1 & ? \\
Perfect sampling using EPCA

Proposition

The algorithm stops a.s. if and only if the EPCA is ergodic.

- If the set of cells is finite, it is the case for positive-rate PCA.
- For $\mathbb{Z}^d$, there exists $\alpha^* \in ]0, 1[$ such that the EPCA is
  - ergodic if $\operatorname{env}(f)(?^N)(?) < \alpha^*$,
  - non-ergodic if $\min_{x \in B^N \setminus A^N} \operatorname{env}(f)(x)(?) > \alpha^*$. 
The majority-flip PCA

\( \alpha = 0.5 \)

\( \alpha = 0.3 \)

Study of the majority-flip PCA

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   - Bernoulli invariant measures
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3. Measures on subshift of finite type
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   - Link with PCA
Back to elementary PCA

\[ E = \mathbb{Z}, \ A = \{0, 1\}, \ \text{neighbourhood of size 2.} \]

\[
\begin{array}{c c c c}
1 & \text{with probability} & \theta_{11} & 1 \\
0 & \text{with probability} & 1 - \theta_{11} & 0 \\
\uparrow & & \uparrow & \\
0 & 0 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c c c c}
1 & \text{with probability} & \theta_{10} & 1 \\
0 & \text{with probability} & 1 - \theta_{10} & 0 \\
\uparrow & & \uparrow & \\
1 & 0 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c c c c}
1 & \text{with probability} & \theta_{01} & 0 \\
0 & \text{with probability} & 1 - \theta_{01} & 1 \\
\uparrow & & \uparrow & \\
0 & 1 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c c c c}
1 & \text{with probability} & \theta_{11} & 1 \\
0 & \text{with probability} & 1 - \theta_{11} & 0 \\
\uparrow & & \uparrow & \\
1 & 1 & 1 & 0 \\
\end{array}
\]
Bernoulli invariant measures

Proposition [Belyaev and al. 1969]

The Bernoulli measure $\mu_p = \mathcal{B}(p)^\otimes \mathbb{Z}$ is an invariant measure of the PCA iff its transitions probabilities satisfy (at least) one of the following equalities.

1. $(1 - p) \cdot \theta_{00} + p \cdot \theta_{01} = (1 - p) \cdot \theta_{10} + p \cdot \theta_{11} = p$

2. $(1 - p) \cdot \theta_{00} + p \cdot \theta_{10} = (1 - p) \cdot \theta_{01} + p \cdot \theta_{11} = p$
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2. $(1 - p) \cdot \theta_{00} + p \cdot \theta_{10} = (1 - p) \cdot \theta_{01} + p \cdot \theta_{11} = p$

Each of these conditions implies surprising properties of the space-time diagrams.

When both conditions are satisfied,

- all the lines of the space-time diagram are i.i.d.

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- the PCA appears in three directions

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- all the lines of the space-time diagram are i.i.d.
- the PCA appears in three directions
- three points are independent unless they form an equilateral triangle pointing up.
Example 1

The choice $\theta_{01} = \theta_{10} = s$ and $\theta_{00} = \theta_{11} = 1 - s$ corresponds to

$$f(x, y) = s \cdot \delta_{x+y \mod 2} + (1 - s) \cdot \delta_{x+y+1 \mod 2}.$$ 

The measure $\mu_{1/2}$ is invariant.

(s=3/4)
Example 2

For every $p \in [0, 1/2]$, one can set

$$\theta_{01} = \theta_{10} = 0, \quad \theta_{11} = 1 \quad \text{and} \quad \theta_{00} = p/(1 - p).$$

This PCA forbids elementary triangles (pointing up) having a single 0.
Larger alphabet

Alphabet \( \mathcal{A} = \{0, \ldots, n\} \).
\( \theta^k_{ij} \) = probability to get \( k \) if the neighbourhood is in state \( ij \).


The Bernoulli measure \( \mu_p \) \( (p = (p_0, \ldots, p_n)) \) is invariant if one of the following conditions is satisfied.

\[
\forall i \in \mathcal{A}, \forall k \in \mathcal{A}, \sum_{j \in \mathcal{A}} p_j \theta^k_{ij} = p_k 
\]

\[
\forall j \in \mathcal{A}, \forall k \in \mathcal{A}, \sum_{i \in \mathcal{A}} p_i \theta^k_{ij} = p_k 
\]

Same properties of space-time diagrams.
Larger alphabet

Alphabet $\mathcal{A} = \{0, \ldots, n\}$.

$\theta_{ij}^k$ = probability to get $k$ if the neighbourhood is in state $ij$.


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\]

Same properties of space-time diagrams.

**Remark:** the deterministic CA that we recover are permutative CA.
The density classification problem

\[ E = \mathbb{Z}^d, \quad A = \{0, 1\} \]

We still denote by \( \mu_p \) the Bernoulli measure of parameter \( p \).

**Challenge**

The density classification problem consists in finding a (P)CA or an IPS \( F \), such that:

\[
\begin{cases}
  p < 1/2 \Rightarrow \mu_p F^{t \to \infty} \to \delta_0, \\
  p > 1/2 \Rightarrow \mu_p F^{t \to \infty} \to \delta_1.
\end{cases}
\]
The density classification problem

\[ E = \mathbb{Z}^d, \ A = \{0, 1\}. \]
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\begin{align*}
    p < 1/2 & \implies \mu_p F^t \xrightarrow{w} \delta_0, \\
    p > 1/2 & \implies \mu_p F^t \xrightarrow{w} \delta_1.
\end{align*}
\]
Ergodicity and perfect sampling
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Measures on subshift of finite type
Bernoulli invariant measures
Density classification

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Measures on subshift of finite type

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PCA and specific measures on symbolic spaces
Definition of Toom’s CA

Toom’s CA is the CA $\mathcal{T}$ on $\mathbb{Z}^2$ of neighborhood $\mathcal{N} = \{ (0,0), (0,1), (1,0) \}$ (north-east-center) defined by the majority rule, that is,

$$(\mathcal{T}(x))_{i,j} = \text{maj}(x_{i,j}, x_{i,j+1}, x_{i+1,j}).$$


Toom’s rule classifies the density.
The proof in pictures
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Ergodicity and perfect sampling
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Bernoulli invariant measures
Density classification

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Steps of the proof

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- Two different 1-clusters cannot merge
- Any finite 1-cluster disappears in finite time and always stays in its enveloping rectangle
- A given point belongs a.s. to the enveloping rectangle of at most finite number of 1-clusters (by the exponential decay of the size of 1-clusters)
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Link with the positive rate problem.
Plan

1. Ergodicity and perfect sampling
   - Undecidability of the ergodicity
   - Perfect sampling

2. PCA having a specific behaviour
   - Bernoulli invariant measures
   - Density classification

3. Measures on subshift of finite type
   - One-dimensional SFT and the Parry measure
   - Link with PCA
Motivation: understanding the combinatorics of multi-dimensional SFT, being able to generate patterns “uniformly”.
**Motivation:** understanding the *combinatorics* of multi-dimensional SFT, being able to *generate patterns “uniformly”*.  

**Example: two-dimensional Fibonacci SFT**  
Set of configurations without two consecutive black squares, vertically or horizontally.
One-dimensional subshift of finite type

Let $\mathcal{A}$ be an alphabet with $n$ letters, and let $\mathcal{A} \in \mathcal{M}_n(\{0, 1\})$. 

One-dimensional subshift of finite type

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**Subshift of finite type**

The *subshift of finite type* associated to $A$ is the set $\Sigma_A$ of words $w \in \mathcal{A}^\mathbb{Z}$ such that if $A_{i,j} = 0$, $w$ does not contain the pattern $ij$.

$$A_{i,j} = \begin{cases} 1 & \text{if } ij \text{ is an allowed pattern}, \\ 0 & \text{if } ij \text{ is a forbidden pattern}. \end{cases}$$

$$\Sigma_A = \{ w \in \mathcal{A}^\mathbb{Z}; \forall k \in \mathbb{Z}, A_{w_k, w_{k+1}} = 1 \}. $$
The Parry measure

Let $\lambda$ be the Perron value of the matrix $A$ (assumed to be irreducible and aperiodic), and let $r$ be the right-eigenvector associated to $\lambda$, satisfying $\sum_{i=1}^{n} r_i = 1$. The Parry measure is the Markov measure $\pi$ of transition matrix $P$ defined, for any $i, j \in \mathcal{A}$, by

$$P_{i,j} = A_{i,j} \frac{r_j}{\lambda r_i}.$$
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The Parry measure is Markov-uniform: for a given $k \geq 1$, the value $\pi(awb)$ does not depend of the word $w \in \{1, \ldots, n\}^k$ such that $awb$ is allowed.
Theorem

Let $\mathcal{M}_{\Sigma_A}$ be the set of translation invariant measures on the SFT $\Sigma_A$, and let $\pi \in \mathcal{M}_{\Sigma_A}$. The following properties are equivalent.

(i) $\pi$ is the Parry measure associated to $\Sigma_A$,
(ii) $\pi$ is a Markov-uniform measure on $\Sigma_A$,
(iii) $\pi$ is the measure of maximal entropy of $\Sigma_A$. 


Example: Fibonacci SFT

Let $\mathcal{A} = \{0, 1\}$. The one-dimensional Fibonacci SFT is the set of words that do not contain two consecutive 1’s. It is given by:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$  

Its Parry measure is the Markov measure given by

$$\pi_0 = \frac{\phi^2}{1 + \phi^2} \quad \text{and} \quad \pi_1 = \frac{1}{1 + \phi^2}.$$
First way to generate the Parry measure

The Parry measure of Fibonacci SFT can be generated by:

- choosing independently to write a 0 with probability \( r_0 = \frac{1}{\phi} \)
  and a 1 with probability \( r_1 = \frac{1}{\phi^2} \),
- rejecting the 1’s creating forbidden patterns.
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Lemma [M. 2013]

For any SFT, the Parry measure can be generated by independent draws of letters, with reject of a letter if it creates a forbidden pattern.
Second way to generate the Parry measure

The Parry measure of Fibonacci SFT can be generated by:

- choosing independently to write a 0 with probability $\tilde{\tau}_0 = \frac{1}{\varphi^2}$
- and a 1 with probability $\tilde{\tau}_1 = \frac{1}{\varphi}$,
- deleting pairs of consecutive 1’s.
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- choosing independently to write a 0 with probability $\tilde{r}_0 = \frac{1}{\varphi^2}$
  and a 1 with probability $\tilde{r}_1 = \frac{1}{\varphi}$,
- deleting pairs of consecutive 1’s.

Proposition [M. 2013]

For confluent SFT, the Parry measure can be generated by independent draws of letters and deletion of forbidden patterns.
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For all $i \in \mathbb{Z}$, if $X_{2i} = X_{2i+2} = 0$, we flip the value of $X_{2i+1}$ with probability $1/2$.

By the Markov-uniform property, the new sequence is still distributed according to $\pi$. 
Link with PCA

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\[ \begin{align*}
\pi &\quad X_{-2} \quad X_{-1} \quad X_0 \quad X_1 \quad X_2 \quad X_3 \quad X_4 \quad X_5 \quad X_6 \\
\pi_2 &\quad X_{-1} \quad X_1 \quad X_3 \quad X_5 \\
F_A &\quad \pi_2 \quad X_{-2} \quad X_0 \quad X_2 \quad X_4 \quad X_6 \\
\end{align*} \]
The projection $\pi_2$ of the Parry measure on odd (resp. even) sites is an invariant measure of the PCA.
Extension

The analogous result holds for any SFT in dimension 1 or in higher dimension.
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Perspectives and works in progress

Perspectives and works in progress
Random walks on free products of groups (with J. Mairesse)

Study of the limit measure

\[ \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/3\mathbb{Z} \]
Random walks on free products of groups (with J. Mairesse)

Generalisation of the Parry measure to SFT defined on infinite trees (with V. Delecroix)
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Deterministic CA and Bernoulli invariant measures

Rigidity and randomisation \textit{(with B. Hellouin de Menibus, A. Maass, M. Sablik)}

\[ f(x, y) = x + y \]
\[ A = \mathbb{Z}/4\mathbb{Z} \]

\[ f(x + y) = \tau_{23}(x + y) \]
- Random walks on free products of groups (with J. Mairesse)
- Generalisation of the Parry measure to SFT defined on infinite trees (with V. Delecroix)
- Deterministic CA and Bernoulli invariant measures
  Rigidity and randomisation (with B. Hellouin de Menibus, A. Maass, M. Sablik)
- Ergodicity of this PCA for large values of $p$. Work related to the study of a “percolation game” (with J. Martin)

\[
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\text{with probability } p \\
\text{with probability } 1 - p
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