KAC-MOODY LIE ALGEBRAS GRADED BY KAC-MOODY ROOT SYSTEMS

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Abstract. We look to gradations of Kac-Moody Lie algebras by Kac-Moody root systems with finite dimensional weight spaces. We extend, to general Kac-Moody Lie algebras, the notion of $C$–admissible pair as introduced by H. Rubenthaler and J. Nervi for semi-simple and affine Lie algebras. If $\mathfrak{g}$ is a Kac-Moody Lie algebra (with Dynkin diagram indexed by $I$) and $(I, J)$ is such a $C$–admissible pair, we construct a $C$–admissible subalgebra $\mathfrak{g}^J$, which is a Kac-Moody Lie algebra of the same type as $\mathfrak{g}$, and whose root system $\Sigma$ grades finitely the Lie algebra $\mathfrak{g}$. For an admissible quotient $\rho: I \to J$ we build also a Kac-Moody subalgebra $\mathfrak{g}^\rho$ which grades finitely the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is affine or hyperbolic, we prove that the classification of the gradations of $\mathfrak{g}$ is equivalent to those of the $C$–admissible pairs and of the admissible quotients.

For general Kac-Moody Lie algebras of indefinite type, the situation may be more complicated; it is (less precisely) described by the concept of generalized $C$–admissible pairs.

2000 Mathematics Subject Classification. 17B67.

Key words and phrases. Kac-Moody algebra, $C$–admissible pair, gradation.

Introduction. The notion of gradation of a Lie algebra $\mathfrak{g}$ by a finite root system $\Sigma$ was introduced by S. Berman and R. Moody [8] and further studied by G. Benkart and E. Zelmanov [5], E. Neher [15], B. Allison, G. Benkart and Y. Gao [1] and J. Nervi [16]. This notion was extended by J. Nervi [17] to the case where $\mathfrak{g}$ is an affine Kac-Moody algebra and $\Sigma$ the (infinite) root system of an affine Kac-Moody algebra; in her two articles she uses the notion of $C$–admissible subalgebra associated to a $C$–admissible pair for the Dynkin diagram, as introduced by H. Rubenthaler [21].

We consider here a general Kac-Moody algebra $\mathfrak{g}$ (indecomposable and symmetrizable) and the root system $\Sigma$ of a Kac-Moody algebra. We say that $\mathfrak{g}$ is finitely $\Sigma$–graded if $\mathfrak{g}$ contains a Kac-Moody subalgebra $\mathfrak{m}$ (the grading subalgebra) whose root system relatively to a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is $\Sigma$ and moreover the action of $ad(\mathfrak{a})$ on $\mathfrak{g}$ is diagonalizable with weights in $\Sigma \cup \{0\}$ and finite dimensional weight spaces, see Definition 1.4. The finite dimensionality of weight spaces is a new condition, it was fulfilled by the non-trivial examples of J. Nervi [17] but it excludes the gradings of infinite dimensional Kac-Moody algebras by finite root systems as in [5]. Many examples of these gradations are provided by the almost split real forms of $\mathfrak{g}$, cf. 1.7. We are interested in describing the possible gradations
of a given Kac-Moody algebra (as in [16], [17]), not in determining all the Lie algebras graded by a given root system \( \Sigma \) (as e.g. in [1] for \( \Sigma \) finite). We carry out completely this project when \( \mathfrak{g} \) is affine or hyperbolic.

Let \( I \) be the index set of the Dynkin diagram of \( \mathfrak{g} \), we generalize the notion of \( C \)-admissible pair \((I, J)\) as introduced by H. Rubenthaler [21] and J. Nervi [16], [17], cf. Definition 2.1. For each Dynkin diagram \( I \) the classification of the \( C \)-admissible pairs \((I, J)\) is easy to deduce from the list of irreducible \( C \)-admissible pairs due to these authors. We are able then to generalize in section 2 their construction of a \( C \)-admissible subalgebra (associated to a \( C \)-admissible pair) which grades finitely \( \mathfrak{g} \):

**Theorem 1.** (cf. 2.6, 2.11, 2.14) Let \( \mathfrak{g} \) be an indecomposable and symmetrizable Kac-Moody algebra, associated to a generalized Cartan matrix \( A = (a_{i,j})_{i,j \in I} \). Let \( J \subset I \) be a subset of finite type such that the pair \((I, J)\) is \( C \)-admissible. There is a generalized Cartan matrix \( A^J = (a^J_{i,k})_{k \in I^J} \) with index set \( I^J = I \setminus J \) and a Kac-Moody subalgebra \( \mathfrak{g}^J \) of \( \mathfrak{g} \) associated to \( A^J \), with root system \( \Delta^J \). Then \( \mathfrak{g} \) is finitely \( \Delta^J \)-graded with grading subalgebra \( \mathfrak{g}^J \).

For a general finite gradation of \( \mathfrak{g} \) with grading subalgebra \( \mathfrak{m} \), we prove (in section 3) that \( \mathfrak{m} \) also is indecomposable, symmetrizable and the restriction to \( \mathfrak{m} \) of the invariant bilinear form of \( \mathfrak{g} \) is non-degenerate (3.11 and 3.17). The Kac-Moody algebras \( \mathfrak{g} \) and \( \mathfrak{m} \) have the same type: finite, affine or indefinite; the first two types correspond to the cases already studied e.g. by J. Nervi. Moreover if \( \mathfrak{g} \) is indefinite Lorentzian or hyperbolic, then so is \( \mathfrak{m} \) (Propositions 3.6 and 3.27). We get also the following precise structure result for this general situation :

**Theorem 2.** Let \( \mathfrak{g} \) be an indecomposable and symmetrizable Kac-Moody algebra, finitely graded by a root system \( \Sigma \) of Kac-Moody type with grading subalgebra \( \mathfrak{m} \).

1) We may choose the Cartan subalgebras \( \mathfrak{a} \) of \( \mathfrak{m} \), \( \mathfrak{h} \) of \( \mathfrak{g} \) such that \( \mathfrak{a} \subset \mathfrak{h} \). Then there is a surjective map \( \rho_a : \Delta \cup \{0\} \to \Sigma \cup \{0\} \) between the corresponding root systems. We may choose the bases \( \Pi_a = \{ \gamma_s \mid s \in I \} \subset \Sigma \) and \( \Pi = \{ \alpha_i \mid i \in I \} \subset \Delta \) of these root systems such that \( \rho_a(\Delta^+ \cap \Sigma^+) \subset \Sigma^+ \cup \{0\} \) and \( \{ \alpha \in \Delta \mid \rho_a(\alpha) = 0 \} = \Delta_J := \Delta \cap \left( \bigcup_{i,j \in I} \mathfrak{a}_{i,j} \right) \) is admissible pair and the situation looks much like the one described by J. Nervi in the finite [16] or affine [17] cases. Actually we prove that this is always true when \( \mathfrak{g} \) is of finite type, affine or hyperbolic (Proposition 3.26). In this real case we get the gradation of \( \mathfrak{g} \) with two levels: \( \mathfrak{g} \) is finitely \( \Delta^J \)-graded with grading subalgebra \( \mathfrak{g}^J \) as in Theorem 1 and \( \mathfrak{g}^J \) is.

It may happen that \( I'_{im} \) is non-empty, we then say that \((I, J)\) is a generalized \( C \)-admissible pair and the gradation is imaginary. We give and explain precisely an example in section 5.

When \( I'_{im} \) is empty (i.e. when the gradation is real : 3.16), \( I_{re} = I \), \( J_{re} = J \), \( \mathfrak{g}(I_{re}) = \mathfrak{g} \), \((I, J) = (I_{re}, J_{re}) \) is a \( C \)-admissible pair and the situation looks much like the one described by J. Nervi in the finite [16] or affine [17] cases. Actually we prove that this is always true when \( \mathfrak{g} \) is of finite type, affine or hyperbolic (Proposition 3.26). In this real case we get the gradation of \( \mathfrak{g} \) with two levels: \( \mathfrak{g} \) is finitely \( \Delta^J \)-graded with grading subalgebra \( \mathfrak{g}^J \) as in Theorem 1 and \( \mathfrak{g}^J \) is.
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finitely $\Sigma$-graded with grading subalgebra $\mathfrak{m}$. But the gradation of $\mathfrak{g}^I$ by $\Sigma$ and $\mathfrak{m}$ is such that the corresponding set "$J$" described as in Theorem 2 is empty; we say (following [16], [17]) that it is a maximal gradation, cf. Definition 3.16 and Proposition 3.21.

To get a complete description of the real gradations, it remains to describe the maximal gradations; this is done in section 4. We prove in Proposition 4.1 that a maximal gradation $(\mathfrak{g}, \Sigma, \mathfrak{m})$ is entirely described by a quotient map $\rho : I \to I$ which is admissible i.e. satisfies two simple conditions (MG1) and (MG2) with respect to the generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$. Conversely for any admissible quotient map $\rho$, it is possible to build a maximal gradation of $\mathfrak{g}$ associated to this map, cf. Proposition 4.5 and Remark 4.7.

1. Preliminaries

We recall the basic results on the structure of Kac-Moody Lie algebras and we set the notations. More details can be found in the book of Kac [12]. We end by the definition of finitely graded Kac-Moody algebras.

1.1. Generalized Cartan matrices. Let $I$ be a finite index set. A matrix $A = (a_{i,j})_{i,j \in I}$ is called a generalized Cartan matrix if it satisfies:

1. $a_{i,i} = 2$  ($i \in I$)
2. $a_{i,j} \in \mathbb{Z}^-$  ($i \neq j$)
3. $a_{i,j} = 0$ implies $a_{j,i} = 0$.

The matrix $A$ is called decomposable if for a suitable permutation of $I$ it takes the form $\left( \begin{array}{cc} B & 0 \\ 0 & C \end{array} \right)$ where $B$ and $C$ are square matrices. If $A$ is not decomposable, it is called indecomposable.

The matrix $A$ is called symmetrizable if there exists an invertible diagonal matrix $D = \text{diag}(d_i, i \in I)$ such that $DA$ is symmetric. The entries $d_i, i \in I$, can be chosen to be positive rationals and if moreover the matrix $A$ is indecomposable, then these entries are unique up to a constant factor.

Any indecomposable generalized Cartan matrix is of one of three mutually exclusive types: finite, affine and indefinite ([12, Chap. 4]). A generalized Cartan matrix is said of finite type if each of its indecomposable factors is of finite type.

An indecomposable and symmetrizable generalized Cartan matrix $A$ is called Lorentzian if it is non-singular and the corresponding symmetric matrix has signature $(+ + ... + -)$; it is then of indefinite type.

An indecomposable generalized Cartan matrix $A$ is called strictly hyperbolic (resp. hyperbolic) if the deletion of any one vertex, and the edges connected to it, of the corresponding Dynkin diagram yields a disjoint union of Dynkin diagrams of finite (resp. finite or affine) type.

Note that a symmetrizable hyperbolic generalized Cartan matrix is non-singular and Lorentzian (cf. [14]).

1.2. Kac-Moody algebras and groups. (See [12] and [18]). Let $A = (a_{i,j})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix. Let $(\mathfrak{h}_\mathbb{R}, \Pi = \{\alpha_i, i \in I\}, \Pi = \{\alpha_i^*, i \in I\})$ be a realization of $A$ over the real field $\mathbb{R}$; thus $\mathfrak{h}_\mathbb{R}$ is a real vector space such that $\dim(\mathfrak{h}_\mathbb{R}) = |I| + \text{corank}(A)$, $\Pi$ and $\Pi^*$ are linearly independent in $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}$ respectively such that $\langle \alpha_j, \alpha_i^* \rangle = a_{i,j}$. Let $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \otimes \mathbb{C}$, then $(\mathfrak{h}, \Pi, \Pi^*)$ is a realization of $A$ over the complex field $\mathbb{C}$.
It follows that, if $A$ is non-singular, then $\Pi$ (resp. $\Pi^*$) is a basis of $\mathfrak{h}$ (resp. $\mathfrak{h}^*$); moreover $\mathfrak{h} = \{ h \in \mathfrak{h} \mid \alpha_i(h) \in \mathbb{R}, \forall i \in I \}$ is well defined by the realization $(\mathfrak{h}, \Pi, \Pi^*)$.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the complex Kac-Moody Lie algebra associated to $A$; it is generated by $\{ h_i, e_i, f_i, i \in I \}$ with the following relations

\begin{align}
\{ [h, h], [e_i, f_j] \} &= 0, & \{ e_i, f_j \} &= \delta_{i,j} \alpha_i^\vee, & (i, j \in I); \\
{[h, e_i]} &= \alpha_i(h) e_i, & {[h, f_i]} &= -\langle \alpha_i, h \rangle f_i & (h \in \mathfrak{h}); \\
\{ \text{ade}_i \}^{1-a_{i,j}}(e_j) &= 0, & \{ \text{ad} f_i \}^{1-a_{i,j}}(f_j) &= 0 & (i \neq j).
\end{align}

The Kac-Moody algebra $\mathfrak{g}'$ of $\mathfrak{g}$ is generated by the Chevalley generators $e_i, f_i, i \in I$, and the center $\mathfrak{c}$ of $\mathfrak{g}$ lies in $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}' = \sum_{i \in I} \mathbb{C} \alpha_i$. If the generalized Cartan matrix $A$ is indecomposable and non-singular, then $\mathfrak{g} = \mathfrak{g}'$ is a (finite or infinite)-dimensional simple Lie algebra, and the center $\mathfrak{c}$ is trivial.

The subalgebra $\mathfrak{h}$ is a maximal ad($\mathfrak{g}$)-diagonalizable subalgebra of $\mathfrak{g}$, it is called the standard Cartan subalgebra of $\mathfrak{g}$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the corresponding root system; then $\Pi$ is a root basis of $\Delta$ and $\Delta = \Delta^+ \cup \Delta^-$, where $\Delta^\pm = \Delta \cap \mathbb{Z}^\pm \Pi$ is the set of positive (or negative) roots relative to the basis $\Pi$. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ corresponding to the root $\alpha$; then $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$.

The Weyl group $W$ of $(\mathfrak{g}, \mathfrak{h})$ is generated by the fundamental reflections $r_i$ ($i \in I$) such that $r_i(h) = h - \langle \alpha_i, h \rangle \alpha_i$ for $h \in \mathfrak{h}$, it is a Coxeter group on $\{ r_i, i \in I \}$ with length function $w \mapsto l(w), w \in W$. The Weyl group $W$ acts on $\mathfrak{h}^*$ and $\Delta$, we set $\Delta^r = W(\Pi)$ (the real roots) and $\Delta^i = \Delta \setminus \Delta^r$ (the imaginary roots). If the generalized Cartan matrix $A$ is indecomposable, then any root basis of $\Delta$ is $W$-conjugate to $\Pi$ or $-\Pi$.

A Borel subalgebra of $\mathfrak{g}$ is a maximal completely solvable subalgebra. A parabolic subalgebra of $\mathfrak{g}$ is a (proper) subalgebra containing a Borel subalgebra. The standard positive (or negative) Borel subalgebra is $\mathfrak{b}^\pm := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$. A parabolic subalgebra $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) containing $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) is called positive (resp. negative) standard parabolic subalgebra of $\mathfrak{g}$; then there exists a subset $J$ of $I$ (called the type of $\mathfrak{p}^\pm$) such that $\mathfrak{p}^\pm = \mathfrak{p}^\pm(J) := \bigoplus_{\alpha \in J} \mathfrak{g}_\alpha$, where $\Delta_J = \Delta \cap \bigoplus_{j \in J} \mathbb{Z} \alpha_j$ (cf. [13]).

In [18], D.H. Peterson and V.G. Kac construct a group $G$, which is the connected and simply connected complex algebraic group associated to $\mathfrak{g}$ when $\mathfrak{g}$ is of finite type, depending only on the derived Lie algebra $\mathfrak{g}'$ and acting on $\mathfrak{g}$ via the adjoint representation $\text{Ad} : G \to \text{Aut}(\mathfrak{g})$. It is generated by the one-parameter subgroups $U_{\alpha} = \exp(\mathfrak{g}_\alpha), \alpha \in \Delta^r$, and $\text{Ad}(U_{\alpha}) = \exp(\text{ad}(\mathfrak{g}_\alpha))$. In the definitions of J. Tits [22] $G$ is the group of complex points of $\mathfrak{G}D$ where $D$ is the datum associated to $A$ and the $\mathbb{Z}$-dual $\Lambda$ of $\bigoplus_{i \in J} \mathbb{Z} \alpha_i^\vee$.

The Cartan subalgebras of $\mathfrak{g}$ are $G$-conjugate. If $\mathfrak{g}$ is indecomposable and not of finite type, there are exactly two conjugate classes (under the adjoint action of $G$) of Borel subalgebras: $G \mathfrak{b}^+$ and $G \mathfrak{b}^-$. A Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ which is $G$-conjugate to $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) is called positive (resp. negative). It follows that
any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is $G$–conjugate to a standard positive (or negative) parabolic subalgebra, in which case, we say that $\mathfrak{p}$ is positive (or negative).

1.3. **Standard Kac-Moody subalgebras and subgroups.** Let $J$ be a non-empty subset of $I$. Consider the generalized Cartan matrix $A_J = (a_{ij})_{i,j \in J}$.

**Definition 1.1.** The subset $J$ is called of finite type if the corresponding generalized Cartan matrix $A_J$ is. We say also that $J$ is connected, if the Dynkin subdiagram, with vertices indexed by $J$, is connected or, equivalently, the corresponding generalized Cartan submatrix $A_J$ is indecomposable.

**Proposition 1.2.** Let $\Pi_J = \{ \alpha_j, \ j \in J \}$ and $\Pi'_J = \{ \alpha'_j, \ j \in J \}$. Let $\mathfrak{h}'_J$ be the subspace of $\mathfrak{h}$ generated by $\Pi_J$, and $\mathfrak{h}'_J' = \Pi'_J = \{ h \in \mathfrak{h}, \langle \alpha_j, h \rangle = 0, \forall j \in J \}$. Let $\mathfrak{h}''_J$ be a supplementary subspace of $\mathfrak{h}'_J + \mathfrak{h}'_J$ in $\mathfrak{h}$ and let

$$\mathfrak{h}_J = \mathfrak{h}'_J \oplus \mathfrak{h}''_J,$$

then, we have:

1) $(\mathfrak{h}_J, \Pi_J, \Pi'_J)$ is a realization of the generalized Cartan matrix $A_J$. Hence $\mathfrak{h}''_J = \{0\}$, $\mathfrak{h}_J = \mathfrak{h}'_J$ when $A_J$ is regular (e.g. when $J$ is of finite type).

2) The subalgebra $\mathfrak{g}(J)$ of $\mathfrak{g}$, generated by $\mathfrak{h}_J$ and the $e_j, f_j, j \in J$, is the Kac-Moody Lie algebra associated to the realization $(\mathfrak{h}_J, \Pi_J, \Pi'_J)$ of $A_J$.

3) The corresponding root system $\Delta(J) = \Delta(\mathfrak{g}(J), \mathfrak{h}_J)$ can be identified with $\Delta_J := \Delta \cap (\oplus_{j \in I} \mathbb{Z}\alpha_j)$.

**N.B.** The derived algebra $\mathfrak{g}'(J)$ of $\mathfrak{g}(J)$ is generated by the $e_j, f_j$ for $j \in J$; it does not depend of the choice of $\mathfrak{h}''_J$.

**Proof.** We may assume $\mathfrak{g}$ indecomposable.

1) Note that $\dim(\mathfrak{h}''_J) = \dim(\mathfrak{h}'_J \cap \mathfrak{h}'_J') = \text{corank}(A_J)$. In particular, $\dim(\mathfrak{h}_J) - |J| = \text{corank}(A_J)$. If $\alpha \in \text{Vect}(\alpha_j, \ j \in J)$, then $\alpha$ is entirely determined by its restriction to $\mathfrak{h}_J$ and hence $\Pi_J$ defines, by restriction, a linearly independent set in $\mathfrak{h}'_J$. As $\Pi_J$ is linearly independent, assertion 1) holds.

Assertions 2) and 3) are straightforward.

In the same way, the subgroup $G_J$ of $G$ generated by $U_{\pm \alpha_j}, \ j \in J$, is equal to the Kac-Moody group associated to the generalized Cartan matrix $A_J$; it is clearly a quotient; the well known equality is proven explicitly in [20, 5.15.2], it may be deduced from [22, th. 1], see also [19, 8.4.2].

1.4. **The invariant bilinear form.** (See [12]).

We recall that the generalized Cartan matrix $A$ is supposed symmetrizable. There exists a non-degenerate $\text{ad}(\mathfrak{g})$– invariant symmetric $\mathbb{C}$–bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$, which is entirely determined by its restriction to $\mathfrak{h}$, such that

$$(\alpha_i, h) = \frac{(\alpha_i, \alpha_i)}{2} (\alpha_i, h), \quad i \in I, \ h \in \mathfrak{h},$$

and we may thus assume that

$$(1.2) \quad (\alpha_i, \alpha_i) \text{ is a positive rational for all } i.$$ 

The non-degenerate invariant bilinear form $(\cdot, \cdot)$ induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ such that $\alpha_i = \frac{2\nu(\alpha_i)}{(\alpha_i, \alpha_i)}$ and $\alpha^*_i = \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i, \alpha_i)}$ for all $i$.

There exists a totally isotropic subspace $\mathfrak{h}''$ of $\mathfrak{h}$ (relative to the invariant bilinear form $(\cdot, \cdot)$).
form $(\langle,\rangle)$ which is in duality with the center $c$ of $\mathfrak{g}$. In particular, $\mathfrak{h}''$ defines a supplementary subspace of $\mathfrak{h}'$ in $\mathfrak{h}$.

Note that any invariant symmetric bilinear form $b$ on $\mathfrak{g}$ satisfying $b(\alpha_i;\alpha_i) > 0$, $\forall i \in I$, is non-degenerate and $b(\alpha_i, h) = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \langle \alpha_i, h \rangle$, $\forall i \in I$, $\forall h \in \mathfrak{h}$. It follows that, if $\mathfrak{g}$ is indecomposable, the restriction of $b$ to $\mathfrak{g}'$ is proportional to that of $(\langle,\rangle)$. In particular, if moreover $A$ is non-singular, then the invariant bilinear form $(\langle,\rangle)$ satisfying the condition 1.2 is unique up to a positive rational factor.

1.5. The Tits cone. (See [12, Chap. 3 and 5]).

Let $C := \{ h \in \mathfrak{h}_R; \langle \alpha_i, h \rangle \geq 0, \forall i \in I \}$ be the fundamental chamber (relative to the root basis $\Pi$) and let $X := \bigcup_{w \in W} w(C)$ be the Tits cone. We have the following description of the Tits cone:

1. $X = \{ h \in \mathfrak{h}_R; \langle \alpha, h \rangle < 0 \text{ only for a finite number of } \alpha \in \Delta^+ \}.$
2. $X = \mathfrak{h}_R$ if and only if the generalized Cartan matrix $A$ is of finite type.
3. If $A$ is indecomposable of affine type, then $X = \{ h \in \mathfrak{h}_R; \langle \delta, h \rangle > 0 \} \cup \mathbb{R} \nu^{-1}(\delta)$, where $\delta$ is the lowest imaginary positive root of $\Delta^+.$
4. If $A$ is indecomposable of indefinite type, then the closure of the Tits cone, for the metric topology on $\mathfrak{h}_R$, is $\bar{X} = \{ h \in \mathfrak{h}_R; \langle \alpha, h \rangle \geq 0, \forall \alpha \in \Delta^+_\text{in} \}.$
5. If $h \in X$, then $h$ lies in the interior $\mathring{X}$ of $X$ if and only if the fixer $W_h$ of $h$, in the Weyl group $W$, is finite. Thus $\mathring{X}$ is the union of finite type facets of $X$.
6. If $A$ is hyperbolic, then $\mathring{X} \cup (-\mathring{X}) = \{ h \in \mathfrak{h}_R; \langle h, h \rangle \leq 0 \}$ and the set of imaginary roots is $\Delta^\text{im} = \{ \alpha \in Q \setminus \{ 0 \}; \langle \alpha, \alpha \rangle \leq 0 \}$, where $Q = \mathbb{Z} \Pi$ is the root lattice.

Remark 1.3. Combining (3) and (4) one obtains that if $A$ is not of finite type then $X = \{ h \in \mathfrak{h}_R; \langle \alpha, h \rangle \geq 0, \forall \alpha \in \Delta^+_\text{in} \}.$


Definition 1.4. Let $\Sigma$ be a root system of Kac-Moody type. The Kac-Moody Lie algebra $\mathfrak{g}$ is said to be finitely $\Sigma$-graded if:

(i) $\mathfrak{g}$ contains, as a subalgebra, a Kac-Moody algebra $\mathfrak{m}$ whose root system relative to a Cartan subalgebra $\mathfrak{a}$ is equal to $\Sigma$.
(ii) $\mathfrak{g} = \sum_{\alpha \in \Sigma \cup \{ 0 \}} V_\alpha$, with $V_\alpha = \{ x \in \mathfrak{g}; [a, x] = \langle \alpha, a \rangle x, \forall a \in \mathfrak{a} \}.$
(iii) $V_\alpha$ is finite dimensional for all $\alpha \in \Sigma \cup \{ 0 \}$.

We say that $\mathfrak{m}$ (as in (i) above) is a grading subalgebra, and $(\mathfrak{g}, \Sigma, \mathfrak{m})$ a gradation with finite multiplicities (or, to be short, a finite gradation).

Note that from (ii) the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is $\text{ad}(\mathfrak{g})$—diagonalizable, and we may assume that $\mathfrak{a}$ is contained in the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Lemma 1.5. Let $\mathfrak{g}$ be a Kac-Moody algebra finitely $\Sigma$-graded, with grading subalgebra $\mathfrak{m}$. If $\mathfrak{m}$ itself is finitely $\Sigma'$-graded (for some root system $\Sigma'$ of Kac-Moody type), then $\mathfrak{g}$ is finitely $\Sigma'$-graded.

Proof. If $\mathfrak{m}'$ is the grading subalgebra of $\mathfrak{m}$, we may suppose the Cartan subalgebras such that $\alpha' \subset \mathfrak{a} \subset \mathfrak{h}$, with obvious notations. Conditions (i) and (ii) are clearly satisfied for $\mathfrak{g}$, $\mathfrak{m}'$ and $\alpha'$. Condition (iii) for $\mathfrak{m}$ and $\Sigma'$ tells that, for all $\alpha' \in \Sigma'$, the set $\{ \alpha \in \Sigma; Q_{\alpha} = \alpha' \}$ is finite. But $V_{\alpha'} = \oplus_{\alpha \alpha' = \alpha'} V_\alpha$, so each $V_{\alpha'}$ is finite dimensional if this is true for each $V_\alpha$.  \( \square \)
1.7. Examples of gradations.

1) Let $\Delta = \Delta(g, h)$ the root system of $g$ relative to $h$, then $g$ is finitely $\Delta$-graded: this is the trivial gradation of $g$ by its own root system.

2) Let $g_R$ be an almost split real form of $g$ (see [2]) and let $t_R$ be a maximal split toral subalgebra of $g_R$. Suppose that the restricted root system $\Delta' = \Delta(g_R, t_R)$ is reduced of Kac-Moody type. In [4, §9], N. Bardy constructed a split real Kac-Moody subalgebra $t_R$ of $g_R$ such that $\Delta' = \Delta(t_R, g_R)$, then $g$ is obviously finitely $\Delta'$-graded.

We get thus many examples coming from known tables for almost split real forms: see [2] in the affine case and [6] in the hyperbolic case.

3) When $g_R$ is an almost compact real form of $g$, the same constructions should lead to gradations by finite root systems, as in [5] e.g.

2. Gradations associated to $C$–admissible pairs.

In this section, we suppose the Kac-Moody Lie algebra $g$ indecomposable and symmetrizable, see however Remark 2.15. We shall build a finite gradation of $g$ associated to some good subset of $I$.

We recall some definitions introduced by H. Rubenthaler ([21]) and J. Nervi ([16], [17]). Let $J$ be a subset of $I$ of finite type. For $k \in I \setminus J$, we denote by $I_k$ the connected component, containing $k$, of the Dynkin subdiagram corresponding to $J \cup \{k\}$, and let $J_k := I_k \setminus \{k\}$.

We are interested in the case where $I_k$ is of finite type for all $k \in I \setminus J$ : that is always true if $g$ is of affine type and $|I \setminus J| \geq 2$ or if $g$ is of hyperbolic type and $|I \setminus J| \geq 3$.

For $k \in I \setminus J$, let $g(I_k)$ be the simple subalgebra generated by $g_{\pm \alpha_i}, i \in I_k$, then $h_{I_k} = h \cap g(I_k) = \sum_{i \in I_k} \mathfrak{c}_i$ is a Cartan subalgebra of $g(I_k)$. Let $H_k$ be the unique element of $h_{I_k}$ such that $\langle \alpha_i, H_k \rangle = 2\delta_{i,k}, \forall i \in I_k$.

**Definition 2.1.** We suppose the Dynkin diagram indexed by $I$ connected and consider a subset $J$ of finite type. We preserve the notations introduced above.

1) Let $k \in I \setminus J$.
   (i) The pair $(I_k, J_k)$ is called admissible if $I_k$ is of finite type and there exist $E_k, F_k \in g(I_k)$ such that $(E_k, H_k, F_k)$ is an $\mathfrak{sl}_2$–triple.
   (ii) The pair $(I_k, J_k)$ is called $C$–admissible if it is admissible and the simple Lie algebra $g(I_k)$ is $A_1$–graded by the root system, of type $A_1$, associated to the $\mathfrak{sl}_2$–triple $(E_k, H_k, F_k)$.

2) The pair $(I, J)$ is called $C$–admissible if the pairs $(I_k, J_k)$ are $C$–admissible for all $k \in I \setminus J$. It is said irreducible if, moreover, $|I \setminus J| = 1$.

Schematically, any $C$–admissible pair $(I, J)$ is represented by the Dynkin diagram, corresponding to $A$, on which the vertices indexed by $J$ are denoted by white circles $\circ$ and those of $I \setminus J$ are denoted by black circles $\bullet$.

**Remark 2.2.** 1) The admissibility of each $(I_k, J_k)$ is essential to build (in 2.6, 2.11) the grading subalgebra $g^J$ and its grading root system $\Delta^J$.

2) As $g(J)$ will be in the eigenspace $V_0$ of weight 0 for the grading by $\Delta^J$, it is necessary to assume $J$ of finite type to get a finite gradation.

3) $I_k$ is of finite type if, and only if, $g(I_k)$ is finite dimensional, and this is equivalent to the alternative assumption in (ii) that the $A_1$–gradation has finite
multiplicities. It is clearly necessary to get, in Theorem 2.14, a finite gradation of \( g \) by the root system \( \Delta^J \). Moreover, even in a more general situation, the condition \( I_k \) of finite type will naturally appear (3.14).

4) Note that the definition presented here, for \( C \)-admissible pairs, is equivalent to that introduced by Rubenthaler and Nervi (see [21], [16]) in terms of prehomogeneous spaces of parabolic type: if \((I_k, J_k)\) is \( C \)-admissible, define for \( p \in \mathbb{Z} \), the subspace \( d_{k,p} := \{ X \in g(I_k) ; [H_k, X] = 2pX \} \); then \((d_{k,0}, d_{k,1})\) is an irreducible regular and commutative prehomogeneous space of parabolic type, and \( d_{k,p} = \{0\} \) for \(|p| \geq 2\). Then \((I_k, J_k)\) is an irreducible \( C \)-admissible pair. According to Rubenthaler and Nervi ([21, Table 1] or [16, Table 2]) the irreducible \( C \)-admissible pair \((I_k, J_k)\) should be among the list in Table 1 below.

5) Along our study of general finite gradations in section 3, we shall meet a situation of "generalized \( C \)-admissible pair" \((I, J)\) (3.16) where \( J \subset I \) is of finite type and \( I_k \) (for \( k \in I' = I \setminus J \)) is defined as above but perhaps not of finite type. When \( k \) is in some subset \( I''_{re} \) of \( I' \), \((I_k, J_k)\) is \( C \)-admissible and the \( k \in I'_{im} = I' \setminus I''_{re} \) do not contribute to the root system \( \Sigma \) grading \( g \). But we do not know the good assumptions on these \((I_k, J_k)\) for \( k \in I'_{im} \) to get, conversely, a finite gradation of \( g \) by some root system. So we give no precise definition; it is expected in the work in preparation [7].

**Table 1**

List of irreducible \( C \)-admissible pairs

<table>
<thead>
<tr>
<th>( A_{2n-1} ), ( n \geq 1 )</th>
<th>( 1 \circlearrowleft 2 \circlearrowleft \cdots \circlearrowleft 2n-1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_n ), ( n \geq 3 )</td>
<td>( 1 \circlearrowleft 2 \circlearrowleft \cdots \circlearrowleft 3 )</td>
</tr>
<tr>
<td>( C_n ), ( n \geq 2 )</td>
<td>( 1 \circlearrowleft 2 \circlearrowleft 3 \circlearrowleft \cdots \circlearrowleft n )</td>
</tr>
<tr>
<td>( D_{2n,1} ), ( n \geq 4 )</td>
<td>( 1 \circlearrowleft 2 \circlearrowleft 3 \circlearrowleft \cdots \circlearrowleft n )</td>
</tr>
<tr>
<td>( D_{2n,2} ), ( n \geq 2 )</td>
<td>( 1 \circlearrowleft 2 \circlearrowleft \cdots \circlearrowleft 2n-1 )</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>( 1 \circlearrowleft 2 \circlearrowleft 3 \circlearrowleft 4 \circlearrowleft 5 \circlearrowleft 6 \circlearrowleft 7 )</td>
</tr>
</tbody>
</table>

**Definition 2.3.** Let \( J \) be a subset of \( I \) and let \( i, k \in I \setminus J \). We say that \( i \) and \( k \) are \( J \)-connected relative to \( A \) if there exist \( j_0, j_1, \ldots, j_{p+1} \in I \) such that \( j_0 = i, j_{p+1} = k \), \( j_s \in J \), \( \forall s = 1, 2, \ldots, p \), and \( a_{j_s, j_{s+1}} \neq 0 \), \( \forall s = 0, 1, \ldots, p \).
Remark 2.4. Note that the relation “to be \(J\)-connected” is symmetric on \(i\) and \(k\). As the generalized Cartan matrix \(A\) is assumed to be indecomposable, for any vertices \(i, k \in I \setminus J\) there exist \(i_0, i_1, \ldots, i_{p+1} \in I \setminus J\) such that \(i_0 = i, i_{p+1} = k\) and \(i_s\) (for \(s = 0, 1, \ldots, p\)) and \(i_s\) and \(i_{s+1}\) are \(J\)-connected.

Let us assume from now on that \((I, J)\) is a \(C\)-admissible pair and let \(I' := I \setminus J\). For \(k \in I'\), let \((E_k, H_k, F_k)\) be an \(\mathfrak{a}_2\)-triple associated to the irreducible \(C\)-admissible pair \((I_k, J_k)\).

Lemma 2.5. Let \(k \neq l \in I'\), then :

1) \(\langle \alpha_i, H_k \rangle \in \mathbb{Z}^-\).
2) The following assertions are equivalent :
   i) \(k, l\) are \(J\)-connected
   ii) \(\langle \alpha_i, H_k \rangle\) is a negative integer
   iii) \(\langle \alpha_k, H_l \rangle\) is a negative integer

Proof. 1) One can write \(H_k = \sum_{i \in I_k} n_{i,k} \alpha_i\), where \(n_{i,k}\) are positive integers (see [21] or [17, 1.4.1.2]). As \(l \notin I_k\), we have that \(\langle \alpha_i, H_k \rangle = \sum_{i \in I_k} n_{i,k} \langle \alpha_i, \alpha_i \rangle \in \mathbb{Z}^-\).

2) In view of Remark 2.4, it suffices to prove the equivalence between i) and ii). Since \(I_k\) is the connected component of \(J \cup \{k\}\) containing \(k\), the assertion i) is equivalent to say that the vertex \(l\) is connected to \(I_k\), so there exists \(i_k \in I_k\) such that \(\langle \alpha_i, \alpha_k \rangle < 0\) and hence \(\langle \alpha_i, H_k \rangle < 0\).

Proposition 2.6. Let \(h' = \Pi_J^I\{h \in h, \langle \alpha_j, h \rangle = 0, \forall j \in J\}\). For \(k \in I'\), denote by \(a'_k\) the restriction of \(\alpha_k\) to the subspace \(h'\) of \(h\), and \(A' = \{a'_k; k \in I'\}\). For \(k, l \in I'\), put \(a'_{k,l} = \langle \alpha_k, H_l \rangle\) and \(A' = \{a'_{k,l}\}_{k,l \in I'}\). Then \(A'\) is an indecomposable and symmetric generalized Cartan matrix, \((h', A', \Pi_J^I)\) is a realization of \(A'\) and \(\text{corank}(A') = \text{corank}(A)\).

Proof. The fact that \(a'_{k,k} = 2\) follows from the definition of \(H_k\) for \(k \in I'\). If \(k \neq l \in I'\), then by Lemma 2.5, \(a'_{k,l} \in \mathbb{Z}^-\) and \(a'_{k,l} \neq 0\) if and only if \(a'_{l,k} \neq 0\). Hence \(A'\) is a generalized Cartan matrix. As the matrix \(A\) is indecomposable, \(A'\) is also indecomposable (see Remark 2.4). Clearly \(\Pi_J^I = \{a'_k; k \in I'\}\) is a linearly independent subset of the dual space \(h''\) of \(h'\), \(\Pi_J^I = \{H_k; k \in I'\}\) is a linearly independent subset of \(h'\) and by construction \(\langle \alpha_i, H_k \rangle = a'_{k,i}, \forall k, l \in I'\).

We have to prove that \(\dim(h') - |I'| = \text{corank}(A')\). As \(J\) is of finite type, the restriction of the invariant bilinear form \((\cdot, \cdot)\) to \(h_J\) is non-degenerate and \(h_J\) is contained in \(h' = \bigoplus_{i \in I} \mathbb{C} \alpha_i\). Therefore

\[
\mathfrak{h} = h' \oplus h_J
\]

and

\[
h' = (h' \cap h') \oplus h_J.
\]

It follows that \(\dim(h' \cap h') = |I'| = \dim(\bigoplus_{k \in I'} \mathbb{C} H_k)\). As the subspace \(\bigoplus_{k \in I'} \mathbb{C} H_k\) is contained in \(h' \cap h'\), we deduce that \(h' \cap h' = \bigoplus_{k \in I'} \mathbb{C} H_k\). Note that any supplementary subspace \(h''\) of \(h' \cap h'\) in \(h'\) is also a supplementary of \(h'\) in \(h\); hence, we have that \(\text{corank}(A) = \dim(h''') = \dim(h') - |I'|\). Let \(\mathfrak{c} := \bigcap_{k \in I'} \ker(\alpha_k)\) be the center of \(\mathfrak{g}\) and let \(\mathfrak{c}' = \bigcap_{k \in I'} \ker(\alpha_k)\). Recall that \(\text{corank}(A) = \dim(\mathfrak{c})\) and
corank\( (A^J) = \dim(c^J) \). It’s clear that \( c^J = c \); hence \( \text{corank}(A^J) = \dim(c^J) = \text{corank}(A) = \dim(h^J) - |I'| \).

It remains to prove that \( A^J \) is symmetrizible. For \( k \in I' \), let \( R_k^J \) be the fundamental reflection of \( h^J \) such that \( R_k^J(h) = h - \langle \alpha_k^l, h \rangle H_k \), \( \forall h \in h^J \). Let \( W^J \) be the Weyl group of \( A^J \) generated by \( R_k^J \), \( k \in I' \). Let \((\ldots,.)^J\) be the restriction to \( h^J \) of the invariant bilinear form \((\ldots,.)\) on \( h \). Then \((\ldots,.)^J\) is a non-degenerate symmetric bilinear form on \( h^J \) which is \( W^J \)-invariant (see the lemma hereafter). From the relation \((R_k^J(H_k),R_k^J(H_l))^J = (H_k,H_l)^J\) one can deduce that:

\[
(H_k,H_l)^J = \frac{(H_k,H_k)^J}{2} a^J_{l,k}, \forall k,l \in I',
\]

but \((H_k,H_k)^J > 0, \forall k \in I'\); hence \( ^tA^J \) (and so \( A^J \)) is symmetrizible. \( \square \)

**Lemma 2.7.** For \( k \in I' := I \setminus J \), let \( w_k^J \) be the longest element of the Weyl group \( W(I_k) \) generated by the fundamental reflections \( r_i, i \in I_k \). Then \( w_k^J \) stabilizes \( h^J \) and induces the fundamental reflection \( R_k^J \) of \( h^J \) associated to \( H_k \).

**Proof.** If one looks at the list above of the irreducible \( C \)-admissible pairs, one can see that \( w_k^J(\alpha_j) = -\alpha_k \) and that \(-w_k^J \) permutes the \( \alpha_j, j \in J_k \). Clearly \( w_k^J(\alpha_j) = \alpha_j, \forall j \in J \setminus J_k \). Hence \( w_k^J \) stabilizes \( h_J \) and its orthogonal subspace \( h_J^J = h^J \). Note that \(-w_k^J(H_k) \in h_k \) and it satisfies the same equations defining \( H_k \). Hence \(-w_k^J(H_k) = H_k = -R_k^J(H_k) \). Recall that \( \ker(\alpha_k^J) = \ker(\alpha_k) \cap (\cap_{j \in J} \ker(\alpha_j)) \); thus it is fixed by \( R_k^J \) and \( W_k^J \). Since \( h^J = \ker(\alpha_k^J) \oplus CH_k \), the reflection \( R_k^J \) coincides with \( W_k^J \) on \( h^J \). \( \square \)

**Remark 2.8.** Actually we can now rediscover the list of irreducible \( C \)-admissible pairs given in Table 1. The black vertex \( k \) should be invariant under \(-w_k^J \) and the corresponding coefficient of the highest root of \( I_k \) should be 1 (an easy consequence of the definition 2.1 (ii) ).

**Example 2.9.** Consider the hyperbolic generalized Cartan matrix \( A \) of type \( HE_8^{(1)} \) = \( E_{10} \) indexed by \( I = \{ -1,0,1,...,8 \} \).

The following two choices for \( J \) define \( C \)-admissible pairs:

1) \( J = \{ 2,3,4,5 \} \).

The corresponding generalized Cartan matrix \( A^J \) is hyperbolic of type \( HF_4^{(1)} \):

2) \( J = \{ 1,2,3,4,5,6 \} \).

The corresponding generalized Cartan matrix \( A^J \) is hyperbolic of type \( HG_2^{(1)} \):

Note that the first example corresponds to an almost split real form of the Kac-Moody Lie algebra \( g(A) \) and \( A^J \) is the generalized Cartan matrix associated to the corresponding (reduced) restricted root system (see [6]) whereas the second example does not correspond to an almost split real form of \( g(A) \).
Lemma 2.10. For $k \in I'$, set $\mathfrak{s}(k) = \mathfrak{CE}_k \oplus \mathfrak{CH}_k \oplus \mathfrak{CF}_k$. Then, the Kac-Moody algebra $\mathfrak{g}$ is an integrable $\mathfrak{s}(k)$—module via the adjoint representation of $\mathfrak{s}(k)$ on $\mathfrak{g}$.

Proof. Note that $\mathfrak{s}(k)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ with standard basis $(\mathfrak{E}_k, \mathfrak{H}_k, \mathfrak{F}_k)$. It is clear that $\text{ad}(H_k)$ is diagonalizable on $\mathfrak{g}$ and $E_k = \sum \alpha, e_\alpha \in d_k, 1$, where $\alpha$ runs over the set $\Delta_{k,1} = \{ \alpha \in \Delta(I_k) : (\alpha, H_k) = 2 \}$, $e_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Delta(I_k)$, and $d_{k,1} := \{ X \in \mathfrak{g}(k) : [H_k, X] = 2X \}$. Since $\Delta_{k,1} \subset \Delta_{g}^{\epsilon}$, $\text{ad}(e_\alpha)$ is locally nilpotent for $\alpha \in \Delta_{k,1}$. As $d_{k,1}$ is commutative (see Remark 2.2) we deduce that $\text{ad}(E_k)$ is locally nilpotent on $\mathfrak{g}$. The same argument shows that $\text{ad}(F_k)$ is also locally nilpotent. Hence, the Kac-Moody algebra $\mathfrak{g}$ is an integrable $\mathfrak{s}(k)$—module.

Proposition 2.11. Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}'$ and $\mathfrak{E}_k,\mathfrak{F}_k, k \in I'$. Then $\mathfrak{g}'$ is the Kac-Moody Lie algebra associated to the realization $(\mathfrak{h}',\Pi',\Pi')$ of the generalized Cartan matrix $A'$. 

Proof. It is not difficult to check that the following relations hold in the Lie subalgebra $\mathfrak{g}'$:

\begin{align*}
[\mathfrak{h}',\mathfrak{h}'] &= 0, \quad [E_k,F_l] = \delta_{k,l}H_k \quad (k, l \in I'); \\
[h,E_k] &= \langle \alpha_k', h \rangle E_k, \quad [h,F_k] = -\langle \alpha_k', h \rangle F_k \quad (h \in \mathfrak{h}', k \in I').
\end{align*}

We have to prove the Serre’s relations :

\((\text{ad}E_k)^{1-a_{k,l}}(E_l) = 0, \quad (\text{ad}F_k)^{1-a_{k,l}}(F_l) = 0 \quad (k \neq l \in I'). \)

For $k \in I'$, let $\mathfrak{s}(k) = \mathfrak{CF}_k \oplus \mathfrak{CH}_k \oplus \mathfrak{CE}_k$ be the Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Let $l \neq k \in I'$; note that $[H_k,F_l] = -a_{k,l}F_l$ and $[E_k,F_l] = 0$, which means that $F_l$ is a primitive weight vector for $\mathfrak{s}(k)$. As $\mathfrak{g}$ is an integrable $\mathfrak{s}(k)$—module (see Lemma 2.10) the primitive weight vector $F_l$ is contained in a finite dimensional $\mathfrak{s}(k)$—submodule (see [12, 3.6]). The relation $(\text{ad}F_k)^{1-a_{k,l}}(F_l) = 0$ follows from the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ (see[12, 3.2]). By similar arguments we prove that $(\text{ad}E_k)^{1-a_{k,l}}(E_l) = 0$.

Now $\mathfrak{g}'$ is a quotient of the Kac-Moody algebra associated to $A'$ and $(\mathfrak{h}',\Pi',\Pi')$. By [12, 1.7] it is equal to it.

Definition 2.12. The Kac-Moody Lie algebra $\mathfrak{g}'$ is called the $C$—admissible algebra associated to the $C$—admissible pair $(I, J)$.

Proposition 2.13. The Kac-Moody algebra $\mathfrak{g}'$ is an integrable $\mathfrak{g}'$—module with finite multiplicities.

Proof. The $\mathfrak{g}'$—module $\mathfrak{g}$ is clearly $\text{ad}(\mathfrak{h}')$—diagonalizable and $\text{ad}(E_k), \text{ad}(F_k)$ are locally nilpotent on $\mathfrak{g}$ for $k \in I'$ (see Lemma 2.10). Hence, $\mathfrak{g}$ is an integrable $\mathfrak{g}'$—module. For $\alpha \in \Delta$, let $\alpha' = \alpha|_{\mathfrak{h}'}$ be the restriction of $\alpha$ to $\mathfrak{h}'$. Set $\Delta' = \{ \alpha' : \alpha \in \Delta \} \setminus \{0 \}$. Then the set of weights, for the $\mathfrak{g}'$—module $\mathfrak{g}$, is exactly $\Delta' \cup \{0 \}$. Note that for $\alpha \in \Delta$, $\alpha' = 0$ if and only if $\alpha \in \Delta(J)$. In particular, the weight space $V_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(J)} \mathfrak{g}_0$ corresponding to the null weight is finite dimensional. Let $\alpha = \sum_{i \in I} n_i \alpha_i \in \Delta$ such that $\alpha' \neq 0$. We will see that the corresponding weight space $V_{\alpha'}$ is finite dimensional. Note that $V_{\alpha'} = \bigoplus_{\beta = \gamma' \Rightarrow \alpha} \mathfrak{g}_\beta$. Let $\beta = \sum_{i \in I} m_i \alpha_i \in \Delta$ such
that $\beta' = \alpha' = \sum_{k \in I'} n_k \alpha_k'$, then $m_k = n_k$, $\forall k \in I'$, since $\Pi^J = \{ \alpha_k', k \in I' \}$ is free in $(\mathfrak{h}^J)^*$. In particular, $\beta'$ and $\alpha$ are of the same sign, and we may assume $\alpha \in \Delta^+$. Let $h_{\gamma, J}(\beta) = \sum_{j \in J} m_j$ be the height of $\beta$ relative to $J$, and let $W_J$ be the finite subgroup of $W$ generated by $r_j$, $j \in J$. Since $W_J$ fixes pointwise $\mathfrak{h}^J$, we deduce that $\gamma' = \beta'$, $\forall \gamma \in W_J \beta$, and so we may assume that $h_{\gamma, J}(\beta)$ is minimal among the roots in $W_J \beta$. From the inequality $h_{\gamma, J}(\beta) \leq h_{\gamma^0, J}(\beta)$, $\forall j \in J$, we get $\langle \beta, \alpha_j \rangle \leq 0$, $\forall j \in J$. Let $\rho_J$ be the half sum of positive coroots of $\Delta(J)$. It is known that $\langle \alpha_j, \rho_J \rangle = 1$, $\forall j \in J$. Note that $\langle \beta, \rho_J \rangle = \sum_{j \in J} m_j + \sum_{k \in I'} n_k \langle \alpha_k, \rho_J \rangle = h_{\gamma, J}(\beta) + \sum_{k \in I'} n_k \langle \alpha_k, \rho_J \rangle$.

Hence, the condition $\langle \beta, \rho_J \rangle \leq 0$ implies $h_{\gamma, J}(\beta) \leq \sum_{k \in I'} -n_k \langle \alpha_k, \rho_J \rangle$. Thus there is just a finite number of possibilities for $\beta$. It follows that $\alpha'$ is of finite multiplicity.

\textbf{Theorem 2.14.} Let $\Delta^J$ be the root system of the pair $(g^J, h^J)$, then the Kac-Moody Lie algebra $g$ is finitely graded, with grading subalgebra $g^J$.

\textbf{Proof.} Let $\Delta' = \{ \alpha', \alpha \in \Delta \} \setminus \{ 0 \}$ be the set of non-null weights of the $g^J$-module $g$ relative to $h^J$. Let $\Delta^J_+ = \{ \alpha' \in \Delta', \alpha \in \Delta^+ \}$ and $\Delta^J_-$ be the set of positive roots of $\Delta^J$ relative to the root basis $\Pi^J$. We have to prove that $\Delta^J = \Delta^J_+ \oplus \Delta^J_-$ or equivalently $\Delta^J_+=\Delta^J_-$. Let $Q^J = \mathbb{Z} \Pi^J$ be the root lattice of $\Delta^J$ and $Q^J_+ = \mathbb{Z}^+ \Pi^J$. It is known that the positive root system $\Delta^J_+$ is uniquely defined by the following properties (see [12, Ex. 5.4]):

(i) $\Pi^J \subset \Delta^J_+ \subset Q^J_+$, $2\alpha' \notin \Delta^J_+$, $\forall i \in I'$;
(ii) if $\alpha' \in \Delta^J_+$, $\alpha' \neq \alpha_i$, then the set $\{ \alpha' + \alpha_i; k \in \mathbb{Z} \} \cap \Delta^J_+$ is a string

\(\{ \alpha' - p \alpha_i, ..., \alpha' + q \alpha_i \}\), where $p, q \in \mathbb{Z}^+$ and $p - q = \langle \alpha', H_i \rangle$;
(iii) if $\alpha' \in \Delta^J_+$, then $\text{supp}(\alpha')$ is connected.

We will see that $\Delta^J_+$ satisfies these three properties and hence $\Delta^J_+ = \Delta^J_-$. Clearly $\Pi^J \subset \Delta^J_+ \subset Q^J_+$. For $\alpha \in \Delta$ and $k \in I'$, the condition $\alpha' \in \Pi_k \alpha_k$ implies $\alpha \in \Delta(I_k)^+$. As $(I_k, J_k)$ is $C$-admissible for $k \in I'$, the highest root of $\Delta(I_k)^+$ has coefficient 1 on the root $\alpha_k$ (cf. Remark 2.8). It follows that $2\alpha' \notin \Delta^J_+$ and (i) is satisfied. By Proposition 2.13, $g$ is an integrable $g^J$-module with finite multiplicities. Hence, the propriety (ii) follows from [12, 3.6]. Let $\alpha \in \Delta_+$, then $\text{supp}(\alpha)$ is connected and $\text{supp}(\alpha') \subset \text{supp}(\alpha)$. Let $k, l \in \text{supp}(\alpha')$; if $k, l$ are $J$-connected in $\text{supp}(\alpha)$ relative to the generalized Cartan matrix $A$ (cf. 2.3), then by lemma 2.5, $k, l$ are linked in $I'$ relative to the generalized Cartan matrix $A^J$. Hence, the connectedness of $\text{supp}(\alpha')$, relative to $A^J$, follows from that of $\text{supp}(\alpha)$ relative to $A$ (see Remark 2.4) and (iii) is satisfied.

\textbf{Remark 2.15.} Note that the definition of $C$-admissible pair can be extended to decomposable Kac-Moody Lie algebras: thus if $I^1, I^2, ..., I^m$ are the connected components of $I$ and $J^k = J \cap I^k$, $k = 1, 2, ..., m$, then $(I, J)$ is $C$-admissible if and only if $(I^k, J^k)$ is for all $k = 1, 2, ..., m$. In particular, the corresponding $C$-admissible algebra is $g^J = \bigoplus_{k=1}^m g(I^k)^{J^k}$, where $g(I^k)^{J^k}$ is the $C$-admissible sub-algebra of $g(I^k)$ corresponding to the $C$-admissible pair $(I^k, J^k)$, $k = 1, 2, ..., m$. 

3. Real gradations.

From now on we suppose that the Kac-Moody Lie algebra $\mathfrak{g}$ is symmetrizable and, starting from 3.5, indecomposable.

Let $\mathfrak{m}$ be a Kac-Moody subalgebra of $\mathfrak{g}$ and let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{m}$. Put $\Sigma = \Delta(\mathfrak{m}, \mathfrak{a})$ the corresponding root system. We assume that $\mathfrak{a} \subset \mathfrak{h}$ and that $\mathfrak{g}$ is finitely $\Sigma$-graded with $\mathfrak{m}$ as grading subalgebra. Thus $\mathfrak{g} = \bigoplus_{\gamma \in \Sigma \cup \{0\}} V_\gamma$, with

$$V_\gamma = \{ x \in \mathfrak{g} : [a,x] = \langle \gamma, a \rangle x, \ \forall a \in \mathfrak{a} \}$$

is finite dimensional for all $\gamma \in \Sigma \cup \{0\}$. For $\alpha \in \Delta$, denote by $\rho_a(\alpha)$ the restriction of $\alpha$ to $\mathfrak{a}$. As $\mathfrak{g}$ is $\Sigma$-graded, one has $\rho_a(\Delta \cup \{0\}) = \Sigma \cup \{0\}$.

**Lemma 3.1.**

1) Let $c$ be the center of $\mathfrak{g}$ and denote by $c_a$ the center of $\mathfrak{m}$. Then $c_a = c \cap \mathfrak{a}$. In particular, if $\mathfrak{g}$ is perfect, then the grading subalgebra $\mathfrak{m}$ is also perfect.

2) Suppose that $\Delta^{im} \neq \emptyset$, then $\rho_a(\Delta^{im}) \subset \Sigma^{im}$.

**Proof.**

1) It is clear that $c \cap \mathfrak{a} \subset c_a$. Since $\mathfrak{g}$ is $\Sigma$-graded, we deduce that $c_a$ is contained in the center $c$ of $\mathfrak{g}$, hence $c_a \subset c \cap \mathfrak{a}$. If $\mathfrak{g}$ is perfect, then $\mathfrak{g} = g'$, $\mathfrak{h} = h'$, $c = \{0\}$; so $c_a = \{0\}$, $a = a'$ and $\mathfrak{m} = \mathfrak{m}'$.

2) If $\alpha \in \Delta^{im}$, then $\mathfrak{N}_a \subset \Delta$. Since $V_0$ is finite dimensional, $\rho_a(\alpha) \neq 0$ and $\mathfrak{N}\rho_a(\alpha) \subset \Sigma$, hence $\rho_a(\alpha) \in \Sigma^{im}$. \hfill $\square$

**Definition 3.2.** ([3, 5.2.6]) Suppose that $\Delta^{im} \neq \emptyset$. Let $\alpha, \beta \in \Delta^{im}$.

(i) The imaginary roots $\alpha$ and $\beta$ are said to be linked if $N\alpha + N\beta \subset \Delta$ or $\beta \in \mathbb{Q}^+ \alpha$.

(ii) The imaginary roots $\alpha$ and $\beta$ are said to be linkable if there exists a finite family of imaginary roots $(\beta_i)_{0 \leq i \leq n+1}$ such that $\beta_0 = \alpha$, $\beta_{n+1} = \beta$ and $\beta_i$ and $\beta_{i+1}$ are linked for all $i = 0, 1, \ldots, n$.

**Proposition 3.3.** ([3, 5.2.7]) Suppose that $\Delta^{im} \neq \emptyset$. Let $\Delta = \bigcup_{j=1}^m \Delta_j$ be the decomposition of $\Delta$ in indecomposable root systems. Suppose that $\Delta_1$, $\Delta_2$, ..., $\Delta_r$ ($r \leq m$) are the indecomposable root subsystems of $\Delta$ which are not of finite type. Then to be linkable is an equivalence relation on $\Delta^{im}$ and the equivalence classes are the $2r$ sets $\Delta^{im}_+ \cap \Delta_j$, $j = 1, 2, \ldots, r$.

**Lemma 3.4.** Suppose that $\Delta^{im} \neq \emptyset$, then there exist root bases in $\Sigma$ and $\Delta$ such that $\rho_a(\Delta^{im}_+) \subset \Sigma^{im}_+$. \hfill $\square$

**Proof.** Fix a root basis $\Pi_a$ for the grading root system $\Sigma$. Let $\Delta = \bigcup_{j=1}^m \Delta_j$ be, as above, the decomposition of $\Delta$ in indecomposable root systems. Denote by $\Pi_j := \Pi \cap \Delta_j$ the root basis of $\Delta_j$, $j = 1, 2, \ldots, m$. If $\alpha, \beta$ are two imaginary linkable roots of $\Delta^{im}_+$, then $\rho_a(\alpha)$ and $\rho_a(\beta)$ are also linkable in $\Sigma^{im}$. By Proposition 3.3, $\rho_a(\alpha)$ and $\rho_a(\beta)$ are of the same sign. Since $\alpha$ and $\beta$ are of the same sign in $\Delta^{im}_j$ relative to the root basis $\Pi_j$, one can, if necessary, change the sign of $\Pi_j$, so that $\rho_a(\alpha)$ and $\rho_a(\beta)$ are positive imaginary roots of $\Sigma^{+}$ relative to the fixed root basis $\Pi_a$. Hence we get a root basis of $\Delta = \bigcup_{j=1}^m \Delta_j$ satisfying $\rho_a(\Delta^{im}_+) \subset \Sigma^{im}_+$. \hfill $\square$

In the following, we will show that the indecomposable Kac-Moody Lie algebra $\mathfrak{g}$ and the grading subalgebra $\mathfrak{m}$ are of the same type.
Lemma 3.5. The Kac-Moody Lie algebra $\mathfrak{g}$ is of indefinite type if and only if $\Delta^{im}$ generates the dual space $(\mathfrak{h}/\mathfrak{c})^*$ of $\mathfrak{h}/\mathfrak{c}$.

Proof. Note that the root basis $\Pi = \{\alpha_i, i \in I\}$ induces a basis for the quotient vector space $(\mathfrak{h}/\mathfrak{c})^*$. It follows that the condition $\Delta^{im} \neq \emptyset$ implies $\dim((\mathfrak{h}/\mathfrak{c})^*) \geq 2$. Suppose now that $\mathfrak{g}$ is of indefinite type. Let $\alpha \in \Delta^{im}$ be a positive strictly imaginary root satisfying $\langle \alpha, \alpha_i^\vee \rangle < 0$, for all $i \in I$. In particular, the vector subspace $\langle \Delta^{im} \rangle$ spanned by $\Delta^{im}$ contains $\Pi$ and hence is equal to $(\mathfrak{h}/\mathfrak{c})^*$. Conversely, if $\Delta^{im}$ generates $(\mathfrak{h}/\mathfrak{c})^*$, then $\Delta^{im}$ is non-empty and contains at least two linearly independent imaginary roots; hence $\Delta$ can not be of finite or affine type. □

Proposition 3.6. 1) If $\Delta^{im}$ is not empty, then $\mathfrak{m}$ is indecomposable.
2) The Kac-Moody Lie Algebra $\mathfrak{g}$ and the grading subalgebra $\mathfrak{m}$ are of the same type.
3) Suppose $\mathfrak{g}$ Lorentzian, then $\mathfrak{m}$ is also Lorentzian.

N.B. We will see below that $\mathfrak{m}$ is always indecomposable (3.11) and symmetrizable (3.17).

Proof. 1) We saw in Lemma 3.4 that $\rho_a(\Delta^{im})$ is in a unique linkable equivalence class of $\Sigma^{im}_+$. So, if $\Sigma = \Sigma_1 \cup \Sigma_2$ is decomposable, we may assume $\rho_a(\Delta^{im}) \subset \Sigma^{im}_1$. But there is $\delta \in \Delta^{im}$ such that $\alpha + n\delta \in \Delta_+$ for all $\alpha \in \Delta_+$ and $n \in \mathbb{N}$ [12, 4.3, 5.6 and 6.3]. So $\rho_a(\alpha) + n\rho_\delta(\delta) \in \Sigma$ for $n >> 0$ and $\rho_a(\alpha) \in \Sigma_1 \cup \{0\}$. As $\rho_a(\Delta \cup \{0\}) = \Sigma \cup \{0\}$, we have $\Sigma_2 = \emptyset$.

2) If $\mathfrak{g}$ is of finite type, then $\Delta$ is finite and hence $\Sigma = \rho_a(\Delta) \setminus \{0\}$ is finite. If $\mathfrak{g}$ is affine, let $\delta$ be the lowest positive imaginary root. One can choose a root basis $\Pi_i = \{\gamma_i, i \in I\}$ of $\Sigma$ so that $\delta := \rho_n(\delta)$ is a positive imaginary root. Note that $a' := a \cap m' \subset h$; in particular $\delta(a') = \{0\}$ and $\langle \delta, \gamma_i \rangle = 0$, for all $i \in I$. It follows that $\mathfrak{m}$ is affine (see [12, 4.3]).

Suppose now that $\mathfrak{g}$ is of indefinite type. Thanks to Lemma 3.5, it suffices to prove that $\Sigma^{im}$ generates $(a/c_a)^*$. where $c_a = c \cap a$ is the center of $\mathfrak{m}$. The natural homomorphism of vector spaces $\pi : a \to \mathfrak{h}/\mathfrak{c}$ induces a monomorphism $\bar{\pi} : a/c_a \to \mathfrak{h}/\mathfrak{c}$. By duality, the homomorphism $\bar{\pi}^* : (\mathfrak{h}/\mathfrak{c})^* \to (a/c_a)^*$ is surjective and $\bar{\pi}^*(\Delta^{im}) \subset \Sigma^{im}$ generates $(a/c_a)^*$.

3) Suppose that $\mathfrak{g}$ is Lorentzian (hence of indefinite type) and let $(\ldots)$ be an invariant non-degenerate bilinear form on $\mathfrak{g}$. Then, the restriction of $(\ldots)$ to $\mathfrak{h}_R$ has signature $(+, +, ..., +, -)$ and any maximal totally isotropic subspace of $\mathfrak{h}_R$ relatively to $(\ldots)$ is one dimensional. Let $a_R := a \cap \mathfrak{h}_R$ and let $(\ldots)_a$ be the restriction of $(\ldots)$ to $\mathfrak{m}$. As $\mathfrak{m}$ is of indefinite type, $\dim(a_R) \geq 2$ and the restriction of $(\ldots)_a$ to $a_R$ is non-null. It follows that the orthogonal subspace $m^\perp$ of $\mathfrak{m}$ relatively to $(\ldots)_a$ is a proper ideal of $\mathfrak{m}$. Since $\mathfrak{m}$ is perfect (because $\mathfrak{g}$ is) we deduce that $m^\perp = \{0\}$ (cf. [12, 1.7]) and the invariant bilinear form $(\ldots)_a$ is non-degenerate. It follows that $\mathfrak{m}$ is symmetrizable and the bilinear form $(\ldots)_a$ when restricted to $a_R$ is non-degenerate; since $\mathfrak{m}$ is of indefinite type, it can not be positive definite. Hence, the bilinear form $(\ldots)_a$ has signature $(+, +, ..., +, -)$ on $a_R$ and then the grading subalgebra $\mathfrak{m}$ is Lorentzian. □

Definition 3.7. Let $\Pi_a$ be a root basis of $\Sigma$ and let $\Sigma^+_a$ be the corresponding set of positive roots. The root basis is said to be adapted to the root basis $\Pi$ of $\Delta$ if $\rho_a(\Delta^+) \subset \Sigma^+_a \cup \{0\}$.

We will see (3.10) that adapted root bases always exist.
Lemma 3.8. Let \( \Pi_a \) be a root basis of \( \Sigma \) such that \( \rho_a(\Delta^m_{im}) \subset \Sigma_{im}^m \) and let \( X_a \) be the corresponding positive Tits cone. Then we have \( \bar{X}_a \subset \bar{X} \cap \mathfrak{a} \).

Proof. As \( \Delta^m_{im} \neq \emptyset \), one has \( \bar{X} = \{ h \in \mathfrak{h}_R; \langle \alpha, h \rangle \geq 0, \forall \alpha \in \Delta^m_{im} \} \) (see Remark 1.3). The lemma follows from Lemma 3.4.

Lemma 3.9. Suppose that \( \Delta^m_{im} \neq \emptyset \). Let \( p \in \bar{X} \) such that \( \langle \alpha, p \rangle \in \mathbb{Z}, \forall \alpha \in \Delta \), and \( \langle \beta, p \rangle > 0, \forall \beta \in \Delta^m_{im} \). Then \( p \in \bar{X} \).

Proof. The result is clear when \( \Delta \) is of affine type since \( \bar{X} = \{ h \in \mathfrak{h}_R; \langle \delta, h \rangle > 0 \} \). Suppose now that \( \Delta \) is of indefinite type. If one looks to the proof of Proposition 5.8.c) in [12], one can show that an element \( p \in X \) satisfying the conditions of the lemma lies in \( X \). As \( \Delta^m_{im} \) is \( W \)-invariant, we may assume that \( p \) lies in the fundamental chamber \( C \). Hence there exists a subset \( J \) of \( I \) such that \( \{ \alpha \in \Delta; \langle \alpha, p \rangle = 0 \} = \Delta_J = \Delta \cap \sum_{j \in J} \mathbb{Z} \alpha_j \). Since \( \Delta_J \cap \Delta^m_{im} = \emptyset \), the root subsystem \( \Delta_J \) is of finite type and \( p \) lies in the finite type facet of type \( J \). Thus \( p \in \bar{X} \) (see 1.5).

Theorem 3.10. There exists a root basis \( \Pi_a \) of \( \Sigma \) which is adapted to the root basis \( \Pi \) of \( \Delta \). Moreover, there exists a finite type subset \( J \) of \( I \) such that \( \Delta_J = \{ \alpha \in \Delta; \rho_a(\alpha) = 0 \} \).

N.B. This is part 1) of Theorem 2.

Proof. Let \( \Pi_a = \{ \gamma_i, i \in \bar{I} \} \) be a root basis of \( \Sigma \) such that \( \rho_a(\Delta^m_{im}) \subset \Sigma_{im}^m \), where \( \bar{I} \) is just a set indexing the basis elements. Let \( p \in \mathfrak{a} \) such that \( \langle \gamma_i, p \rangle = 1, \forall i \in \bar{I} \) and let \( P = \{ \alpha \in \Delta; \langle \alpha, p \rangle \geq 0 \} \). If \( \Delta \) is finite, then \( P \) is clearly a parabolic subsystem of \( \Delta \) and the result is trivial. Suppose now that \( \Delta^m_{im} \neq \emptyset \); then \( p \) satisfies the conditions of the Lemma 3.9 and we may assume that \( p \) lies in the facet of type \( J \) for some subset \( P = \Delta_J \cup \Delta^+ \) is the standard parabolic subsystem of finite type \( J \). Note that, for \( \gamma \in \Sigma^+, \) one has \( \gamma = h_{\rho_a}(\gamma) \) the height of \( \gamma \) with respect to \( \Pi_a \). It follows that \( \{ \alpha \in \Delta; \rho_a(\alpha) = 0 \} = \Delta_J \), in particular, \( \rho_a(\Delta^+) = \rho_a(P) \subset \Sigma^+ \cup \{ 0 \} \). Hence, the root basis \( \Pi_a \) is adapted to \( \Pi \).

Corollary 3.11. \( \Sigma \) is indecomposable.

Proof. For \( \gamma_1, \gamma_2 \in \Pi_a \), there are \( \alpha_1, \alpha_2 \in \Delta_+ \) such that \( \gamma_i = \rho_a(\alpha_i) \). But \( \gamma_i \) is not a sum in \( \Sigma^+ \), so, up to \( \Delta_J, \alpha_i \) is not a sum: we may assume \( \alpha_i \in \Pi \). As \( \Delta \) is indecomposable, there is a root \( \alpha \in \Delta \cap (\alpha_1 + \alpha_2 + \sum_{\alpha \in \Pi} \mathbb{Z}^+ \alpha) \). Now \( \rho_a(\alpha) \in (\Sigma \cup \{ 0 \}) \cap (\gamma_1 + \gamma_2 + \sum_{\gamma \in \Pi_a} \mathbb{Z}^+ \gamma) \subset \Sigma \) and \( \gamma_1, \gamma_2 \) have to be in the same connected component of \( \Pi_a \).

From now on, we fix a root basis \( \Pi_a = \{ \gamma_s, s \in I \} \), for the grading root system \( \Sigma \), which is adapted to the root basis \( \Pi = \{ \alpha_i, i \in I \} \) of \( \Delta \) (see Theorem 3.10). As before, let \( J := \{ j \in I; \rho_a(\alpha_j) = 0 \} \) and \( J' := I \setminus J \). For \( k \in I' \), we denote, as above, by \( I_k \) the connected component of \( J \cup \{ k \} \) containing \( k \), and \( J_k := J \cap I_k \).

Proposition 3.12.

1) Let \( s \in I \), then there exists \( k_s \in I' \) such that \( \rho_a(\alpha_{k_s}) = \gamma_s \) and any preimage \( \alpha \in \Delta \) of \( \gamma_s \) is equal to \( \alpha_k \) modulo \( \sum_{j \in J_k} \mathbb{Z} \alpha_j \) for some \( k \in I' \) satisfying \( \rho_a(\alpha_k) = \gamma_s \).
2) Let \( k \in I' \) such that \( \rho_a(\alpha_k) \) is a real root of \( \Sigma \). Then \( \rho_a(\alpha_k) \in \Pi_a \) is a simple root.

Proof. This result was proved by J. Nervi for affine algebras (see [17, 2.3.10] and the proof of Prop. 2.3.12). The arguments used there are available for general Kac-Moody algebras.

We introduce the following notations:
\[
I'_{\text{re}} := \{ i \in I' ; \rho_a(\alpha_i) \in \Pi_a \}; \quad I'_{\text{im}} := I' \setminus I'_{\text{re}},
\]
\[
I_{\text{re}} = \bigcup_{k \in I'_{\text{re}}} I_k; \quad J_{\text{re}} = I_{\text{re}} \cap J = \bigcup_{k \in I'_{\text{re}}} J_k; \quad J^0 = J \setminus J_{\text{re}}
\]
\[
\Gamma_s := \{ i \in I' ; \rho_a(\alpha_i) = \gamma_s \}, \forall s \in I.
\]

Note that \( J^0 \) is not connected to \( I_{\text{re}} \).

Remark 3.13.
1) In view of Proposition 3.12, assertion 2), one has \( \rho_a(\alpha_k) \in \Sigma^\text{im}_+ \), \( \forall k \in I'_{\text{im}} \).
2) \( I = I_{\text{re}} \cup I_{\text{im}} \cup J^0 \) is a disjoint union.
3) If \( I'_{\text{im}} = \emptyset \), then \( I = I_{\text{re}} \cup J^0 \). Since \( I \) is connected (and \( I_{\text{re}} \) is not connected to \( J^0 \)) we deduce that \( J^0 = \emptyset \), \( I = I_{\text{re}} \) and \( I'_{\text{re}} = I' = I \setminus J \).
4) If \( I'_{\text{im}} \neq \emptyset \), then \( I_{\text{re}} \) may be non-connected (see the example in §5 below).

1) Let \( k \in I'_{\text{re}} \), then \( I_k \) is of finite type.
2) Let \( s \in I \). If \( |\Gamma_s| \geq 2 \) and \( k \neq l \in \Gamma_s \), then \( I_k \cup I_l \) is not connected: \( g(I_k) \) and \( g(I_l) \) commute and are orthogonal.
3) For all \( k \in I'_{\text{re}}, (I_k, J_k) \) is an irreducible \( C \)–admissible pair.
4) The derived subalgebra \( m' \) of the grading algebra \( m \) is contained in \( g'(I_{\text{re}}) \) (as defined in proposition 1.2).

Proof.
1) Suppose that there exists \( k \in I'_{\text{re}} \) such that \( I_k \) is not of finite type; then there exists an imaginary root \( \beta_k \) whose support is the whole \( I_k \). Hence, there exists a positive integer \( m_k \in \mathbb{N} \) such that \( \rho_a(\beta_k) = m_k \rho_a(\alpha_k) \) is an imaginary root of \( \Sigma \). It follows that \( \rho_a(\alpha_k) \in \Sigma^\text{im} \) is an imaginary root and this contradicts the fact that \( k \in I'_{\text{re}} \).
2) Let \( s \in I \) such that \( |\Gamma_s| \geq 2 \) and \( k \neq l \in \Gamma_s \). Since \( V_{\gamma_{ls}} = \{0\} \) for all integer \( n \geq 2 \), the same argument used in 1) shows that \( I_k \cup I_l \) is not connected, and \( I_k \) and \( I_l \) are its two connected components. In particular, \( [g(I_k),g(I_l)] = \{0\} \) and \( (g(I_k),g(I_l)) = \{0\} \).
3) Let \( k \in I'_{\text{re}} \) and let \( s \in I \) such that \( \rho_a(\alpha_k) = \gamma_s \). Let \( (\tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s) \) be an \( \mathfrak{sl}_2 \)–triple in \( m \) corresponding to the simple root \( \gamma_s \). Let \( V_{\gamma_s} \) be the weight space of \( m \) corresponding to \( \gamma_s \). In view of Proposition 3.12, assertion 1), one has:
\[
V_{\gamma_s} = \bigoplus_{l \in \Gamma_s} V_{\gamma_s} \cap g(I_l).
\]
Hence, one can write:
\[
\tilde{X}_s = \sum_{l \in \Gamma_s} E_l; \quad \tilde{Y}_s = \sum_{l \in \Gamma_s} F_l,
\]
with \( E_l \in V_{\gamma_s} \cap g(I_l) \) and \( F_l \in V_{-\gamma_s} \cap g(I_l) \). It follows from assertion 2) that
\[
\tilde{H}_s = \gamma_s = [\tilde{X}_s, \tilde{Y}_s] = \sum_{l \in \Gamma_s} [E_l, F_l] = \sum_{l \in \Gamma_s} H_l,
\]
where \( H_l := [E_l, F_l] \in \mathfrak{h}_I, \forall l \in \Gamma_s \). Then one has, for \( k \in \Gamma_s \),

\[
2 = \langle \gamma_s, \gamma_s^* \rangle = \langle \alpha_k, \gamma_s^* \rangle = \sum_{l \in \Gamma_s} \langle \alpha_k, H_l \rangle = \langle \alpha_k, H_k \rangle,
\]

and for \( j \in J_k \),

\[
0 = \langle \alpha_j, \gamma_s^* \rangle = \sum_{l \in \Gamma_s} \langle \alpha_j, H_l \rangle = \langle \alpha_j, H_k \rangle.
\]

In particular, \( H_k \) is the unique semi-simple element of \( \mathfrak{h}^I \) satisfying :

\[
(\alpha_i, H_k) = 2\delta_{i,k}, \forall i \in I_k.
\]

Hence, \((E_k, H_k, F_k)\) is an \( \mathfrak{sl}_2 \)–triple in the simple Lie algebra \( \mathfrak{g}(I_k) \) and since \( V_{2\gamma_s} = \{0\} \), \((I_k, J_k)\) is an irreducible \( C \)–admissible pair for all \( k \in \Gamma_s \). The statement 4) follows from the relation (3.2). \( \square \)

**Corollary 3.15.** The pair \((I_{re}, J_{re})\) is \( C \)–admissible (in the eventually decomposable sense of Remark 2.15). If \( I_{im}' = \emptyset \), then \( I_{re} = I \), \( J_{re} = J \) and \( \mathfrak{g} \) is finitely \( \Delta^J \)–graded, with grading subalgebra \( \mathfrak{g}' \).

**N.B.** We have got part 2) of Theorem 2.

**Proof.** The first assertion is a consequence of Proposition 3.14. By remark 3.13, when \( I_{im}' = \emptyset \), we have \( I = I_{re} \); hence, by Theorem 2.14, \( \mathfrak{g} \) is finitely \( \Delta^J \)–graded. \( \square \)

**Definition 3.16.** If \( I_{im}' \neq \emptyset \), then \((I, J)\) is called a generalized \( C \)–admissible pair and the gradation of \( \mathfrak{g} \) by \( \Sigma \) and \( \mathfrak{m} \) is said imaginary.

On the contrary if \( I_{im}' = \emptyset \), the gradation is said real.

If \( I_{im}' = J = \emptyset \), the Kac-Moody algebra \( \mathfrak{g} \) is said to be maximally finitely \( \Sigma \)–graded.

**Corollary 3.17.** The grading subalgebra \( \mathfrak{m} \) of \( \mathfrak{g} \) is symmetrizable and the restriction to \( \mathfrak{m} \) of the invariant bilinear form of \( \mathfrak{g} \) is non-degenerate.

**Proof.** Let \((\ldots)_a \) be the restriction to \( \mathfrak{m} \) of the invariant bilinear form \((\ldots)\) of \( \mathfrak{g} \).

Recall from the proof of Proposition 3.14 that \( \gamma_s^* = \sum_{k \in \Gamma_s} H_k, \forall s \in \bar{I} \). In particular \((\gamma_s^*, \gamma_s^*)_a = \sum_{k \in \Gamma_s} (H_k, H_k) > 0 \).

It follows that \((\ldots)_a \) is a non-degenerate invariant bilinear form on \( \mathfrak{m} \) (see §1.4) and that \( \mathfrak{m} \) is symmetrizable. \( \square \)

**Corollary 3.18.** Let \( \mathfrak{h}^J \) be the orthogonal of \( \mathfrak{h}_J \) in \( \mathfrak{h} \). For \( k \in I_{im}' \), write

\[
\rho_a(\alpha_k) = \sum_{s \in \bar{I}} n_{s,k} \gamma_s.
\]

For \( s \in \bar{I} \), choose \( l_s \) a representative element of \( \Gamma_s \). Then \( \mathfrak{a}/\mathfrak{c}_a \) can be viewed as the subspace of \( \mathfrak{h}^J/\mathfrak{c} \) defined by the following relations :

\[
\langle \alpha_k, h \rangle = \langle \alpha_l, h \rangle, \forall k \in \Gamma_s, \forall s \in \bar{I}
\]

\[
\langle \alpha_k, h \rangle = \sum_{s \in \bar{I}} n_{s,k} \langle \alpha_s, h \rangle, \forall k \in I_{im}'.
\]

**Proof.** The subspace of \( \mathfrak{h}^J/\mathfrak{c} \) defined by the above relations has dimension \(|\bar{I}|\) and contains \( \mathfrak{a}/\mathfrak{c}_a \) and hence it is equal to \( \mathfrak{a}/\mathfrak{c}_a \). \( \square \)
Proposition 3.19. Let $(\ldots)_a$ be the restriction to $\mathfrak{m}$ of the invariant bilinear form $(\ldots)$ of $\mathfrak{g}$.

1) Let $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$ and let $\mathfrak{a}''$ be a supplementary subspace of $\mathfrak{a}'$ in $\mathfrak{a}$ which is totally isotropic relatively to $(\ldots)_a$. Then $\mathfrak{a}'' \cap \mathfrak{h}' = \{0\}$.

2) Let $A_{I_{re}}$ be the submatrix of $A$ indexed by $I_{re}$. Then there exists a subspace $\mathfrak{h}_{I_{re}}$ of $\mathfrak{h}$ containing $a$ such that $(\mathfrak{h}_{I_{re}}, \Pi_{I_{re}}, \Pi_{I_{re}}')$ is a realization of $A_{I_{re}}$. In particular, the Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ associated to this realization (in 1.2) contains the grading subalgebra $\mathfrak{m}$.

3) The Kac-Moody algebra $\mathfrak{g}(I_{re})$ is finitely $\Delta(I_{re})^{J_{re}}$-graded and its grading subalgebra is the subalgebra $\mathfrak{g}(I_{re})^{J_{re}}$ associated to the $C$-admissible pair $(I_{re}, J_{re})$ as in Proposition 2.11.

4) The Kac-Moody algebra $\mathfrak{g}(I_{re})^{J_{re}}$ contains $\mathfrak{m}$.

Proof. 

1) Recall that the center $\mathfrak{c}_a$ of $\mathfrak{m}$ is contained in the center $\mathfrak{c}$ of $\mathfrak{g}$. Since $\mathfrak{h}' = \mathfrak{c}_a$ and $\mathfrak{c}_a$ is in duality with $\mathfrak{a}''$ relatively to $(\ldots)_a$, we deduce that $\mathfrak{a}'' \cap \mathfrak{h}' = \{0\}$.

2) From the proofs of 3.17 and 3.14 we get $\gamma^\vee = \sum_{k \in I_{re}} H_k \in \sum_{k \in I_{re}} \mathfrak{h}_k = \mathfrak{h}_{I_{re}}$. So $\mathfrak{c}_a \subset \mathfrak{a}' \subset \mathfrak{h}_{I_{re}} \subset \mathfrak{h}'$. It follows that $(\mathfrak{h}_{I_{re}}' + \mathfrak{h}'^{I_{re}})$ is contained in $\mathfrak{c}_a$ the orthogonal subspace of $\mathfrak{c}_a$. Since $\mathfrak{a}'' \cap \mathfrak{c}_a = \{0\}$, one can choose a supplementary subspace $\mathfrak{h}_{I_{re}}''$ of $(\mathfrak{h}_{I_{re}}' + \mathfrak{h}'^{I_{re}})$ containing $\mathfrak{a}''$. Let $\mathfrak{h}_{I_{re}} = \mathfrak{h}_{I_{re}}' \oplus \mathfrak{h}_{I_{re}}''$. Then, by Proposition 1.2, $(\mathfrak{h}_{I_{re}}, \Pi_{I_{re}}, \Pi_{I_{re}}')$ is a realization of $A_{I_{re}}$.

3) As in Corollary 3.15, assertion 3) is a simple consequence of Theorem 2.14.

4) The algebra $\mathfrak{a}$ is in $\mathfrak{h}_{I_{re}} \cap \Pi_{I_{re}}' = (\mathfrak{h}_{I_{re}})^{J_{re}}$. By the proof of Proposition 3.14, for $s \in I$, $X_s$ and $Y_s$ are linear combinations of the elements in $\{E_k, F_k \mid k \in \Gamma_s\} \subset \mathfrak{g}(I_{re})^{J_{re}}$. Hence $\mathfrak{g}(I_{re})^{J_{re}}$ contains all generators of $\mathfrak{m}$. \hfill $\Box$

Lemma 3.20. Let $\mathfrak{r}$ be a Kac-Moody subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}$. Then $\mathfrak{r}$ is finitely $\Sigma$-graded. In particular, the Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ or $\mathfrak{g}(I_{re})^{J_{re}}$ is finitely $\Sigma$-graded.

N.B. Proposition 3.19 and Lemma 3.20 finish the proof of Theorem 2.

Proof. Recall that the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is ad$_\mathfrak{g}$-diagonalizable. Since $\mathfrak{r}$ is ad($\mathfrak{a}$)-invariant, one has $\mathfrak{r} = \sum_{\gamma \in \Sigma} V_\gamma \cap \mathfrak{r}$. By assumption $\{0\} \neq \mathfrak{m} \subset V_\gamma \cap \mathfrak{r}$ for all $\gamma \in \Sigma$. Thus, $\mathfrak{r}$ is finitely $\Sigma$-graded. \hfill $\Box$

Proposition 3.21. If $I_{re}' = \emptyset$, then $\mathfrak{g}(I_{re}) = \mathfrak{g}$ and the $C$-admissible subalgebra $\mathfrak{g}^J$ is maximally finitely $\Sigma$-graded, with grading subalgebra $\mathfrak{m}$.

Proof. This result is due to J. Nervi ([17, 2.5.10]) for the affine case; it follows from the facts that $V_0 \cap \mathfrak{g}^J = \mathfrak{h}^J$ and $\mathfrak{m} \subset \mathfrak{g}^J$ (see Prop. 3.19). \hfill $\Box$

We now want a precise description of the gradation of $\mathfrak{g}(I_{re})$ by $\Sigma$ and $\mathfrak{m}$; particularly in the case (already mentioned in Remark 3.13) where $\mathfrak{g}(I_{re})$ (and so $\mathfrak{g}(I_{re})^{J_{re}}$) is decomposable.

Let $I_{I_{re}}^1, I_{I_{re}}^2, \ldots, I_{I_{re}}^q$ be the connected components of $I_{re}$ and $J_{I_{re}}^i := I_{re} \cap I_{I_{re}}^i$, $i = 1, 2, \ldots, q$. Then $\mathfrak{g}(I_{re}) = \bigoplus_{i=1}^q \mathfrak{g}(I_{I_{re}}^i)$ and hence $\mathfrak{g}(I_{re})^{J_{re}} = \bigoplus_{i=1}^q \mathfrak{g}(I_{I_{re}}^i)^{J_{re}}$ (see Remark 2.15). Retain the notations introduced just before Proposition 3.14 and
those introduced in its proof.

For $s \in \bar{I}$ and $i = 1, 2, \ldots, q$, let $\Gamma^i_s := \Gamma_s \cap I^i_{re}$. If $\Gamma^i_s$ is non-empty, put $E^i_s := \sum_{l \in \Gamma^i_s} E_l$;

$$
F^i_s := \sum_{l \in \Gamma^i_s} F_l \quad \text{and} \quad H^i_s := \sum_{l \in \Gamma^i_s} H_l.
$$

We take $E^i_s = F^i_s = H^i_s = 0$ if $\Gamma^i_s$ is empty. Note that $\Gamma^i_s = \bigcup_{i=1}^q \Gamma^i_s$ (disjoint union) and from the proof of the Proposition 3.14 we get the following relations

$$
X_s = \sum_{i=1}^q E^i_s; \quad Y_s = \sum_{i=1}^q F^i_s, \forall s \in \bar{I},
$$

(3.5)

$$
\bar{H}_s = \gamma_s = [X_s, Y_s] = \sum_{i=1}^q [E^i_s, F^i_s] = \sum_{i=1}^q H^i_s, \forall s \in \bar{I}.
$$

(3.6)

Lemma 3.22. Let $s \in \bar{I}$ and $i \in \{1, 2, \ldots, q\}$ such that $\Gamma^i_s \neq \emptyset$. Then we have

1) $\Gamma^i_s \neq \emptyset$ for all $t \in \bar{I}$ satisfying $\langle \gamma_t, \gamma_s \rangle < 0$.

2) $\Gamma^i_s \neq \emptyset, \forall t \in \bar{I}$.

Proof. To prove 1), suppose $\Gamma^i_s = \emptyset$ for any $t$ satisfying $\langle \gamma_t, \gamma_s \rangle < 0$. Let $k \in \Gamma^i_s$, then $\langle \gamma_s, \gamma_t \rangle = \sum_{j=1}^q \langle \alpha_k, H^j_l \rangle = 0$, a contradiction since $\langle \gamma_s, \gamma_t \rangle$ must be negative.

Thus $\Gamma^i_s \neq \emptyset$ iff $\Gamma^i_s \neq \emptyset$. The second statement follows from the connectedness of $\bar{I}$:

For $t \in \bar{I}$, there exists a sequence $s_0 = s, s_1, \ldots, s_n = t$ in $\bar{I}$ such that $s_j$ is linked to $s_{j+1}$ for all $j = 0, 1, \ldots, n - 1$. By 1) $\Gamma^i_{s_{j+1}}$ is, as $\Gamma^i_s$, non-empty for all $j = 0, 1, \ldots, n$.

In particular $\Gamma^i_t \neq \emptyset$. □

Lemma 3.23. $\Gamma^i_s \neq \emptyset, \forall s \in \bar{I}, \forall i = 1, 2, \ldots, q$, and $(H^i_s)_{s \in \bar{I}}$ is free for all $i = 1, 2, \ldots, q$.

Proof. Recall that $I_{re} = \bigcup_{k \in I_{re}} I_k$, with all the $I_k$ connected. Let $i \in \{1, 2, \ldots, q\}$ and let $k \in I^i_{re}$ such that $I_k \subset I^i_{re}$. Let $s \in \bar{I}$ such that $\rho_0(\alpha_k) = \gamma_s$, then $k \in \Gamma^i_s$ and $\Gamma^i_s \neq \emptyset$. By the Lemma 3.22, $\Gamma^i_s \neq \emptyset$ for all $t \in \bar{I}$. Thus $H^i_s \neq 0, \forall s \in \bar{I}; \forall i = 1, 2, \ldots, q$, and the freeness of $(H^i_s)_{s \in \bar{I}}$ follows from that of $(H^i_k)_{k \in I_{re}}$. □

Proposition 3.24. For $i = 1, 2, \ldots, q$, let $p_i$ be the projection of $g(I_{re})$ on $g(I^i_{re})$ with kernel $\bigoplus_{j \neq i} g(I^j_{re})$ and let $m_i := p_i(m)$. Then we have:

1) $m_i$ is a Kac-Moody subalgebra of $g(I^i_{re})$ isomorphic to $m$.

2) The Kac-Moody subalgebra $g(I^i_{re})$ is maximally finitely $\Sigma_i$-graded, where $\Sigma_i$ is the root system of $m_i$ relative to the Cartan subalgebra $a_i := p_i(a)$.

N.B. Note that $m$ is contained in $\bigoplus_{i=1}^q m_i$. In particular, $\bigoplus_{i=1}^q m_i$ is finitely $\Sigma$-graded.

If we identify $\bigoplus_{i=1}^q m_i$ with $m^\emptyset$, then the grading subalgebra $m$ can be viewed as the diagonal subalgebra $\Delta(m^\emptyset)$ of $m^\emptyset$: $\Delta(m^\emptyset) := \{(X, X, \ldots, X) : X \in m\}$.

Proof. For $i \in \{1, 2, \ldots, q\}$, $p_i$ is a morphism of Lie algebras and $m_i := p_i(m)$ is contained in $g(I^i_{re})$ isomorphic to $m$. For $s \in I$, one has $p_i(\gamma_s) = H^i_s$. Thus the restriction of $p_i$ to $a^\emptyset := [a, a] = \bigoplus_{s \in I} C \gamma_s^\emptyset$ is injective by Lemma 3.23. Since $m$ is indecomposable, $p_i$
when restricted to \( m \) is still injective (see [12, 1.7]). Thus \( m_i = p_i(m) \) is isomorphic to \( m \) and we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{m} & \xrightarrow{\sim} & \text{m}_i \\
\downarrow & & \downarrow \\
\mathfrak{g}(I_{rc}) & \xrightarrow{p_i} & \mathfrak{g}(I_{rc}^i)
\end{array}
\]

For the second assertion, Let \( a_i := p_i(a) \) and \( \Sigma_i = \Delta(m_i, a_i) \). When restricted to \( m \)
\( p_i \) induces an isomorphism of root systems \( \psi_i : \Sigma_i \to \Sigma \) such that
\[
\langle \alpha, h \rangle = \langle \psi_i^{-1}(\alpha), p_i(h) \rangle, \quad \forall \alpha \in \Sigma, \forall h \in a.
\]

Note that for \( \alpha \in \Sigma \) and \( X \in \mathfrak{g}(I_{rc}) \) satisfying \( [h, X] = \langle \alpha, h \rangle X \), \( \forall h \in a \), one has
\[
[h, p_i(X)] = \langle \psi_i^{-1}(\alpha), h' \rangle p_i(X), \quad \forall h' \in a_i.
\]

Since \( \mathfrak{g}(I_{rc}) \) (resp. \( \mathfrak{g}(I_{rc}^i) \)) is finitely \( \Sigma \)-graded and \( p_i \) is surjective, the Kac-Moody subalgebra \( \mathfrak{g}(I_{rc}^i) \) (resp. \( \mathfrak{g}(I_{rc})^i \))
is finitely \( \Sigma_i \)-graded. For \( k \in I_{rc} \), Let \( \rho_i(\alpha_k) \) be the restriction of \( \alpha_k \) to \( a_i \). Then
\[
(\rho_i(\alpha_k) = 0) \iff (\rho_a(\alpha_k) = 0) \iff (k \in I_{rc}).
\]

By Proposition 3.21, \( \mathfrak{g}(I_{rc})^i \) is maximally finitely \( \Sigma_i \)-graded.

**Corollary 3.25.** If \( g \) is Lorentzian then \( I_{rc} \) is connected.

**Proof.** If \( g \) is Lorentzian, then By Proposition 3.6, the grading subalgebra \( m \) and hence all the \( m_i \) \( (i = 1, 2, \ldots, q) \) are also Lorentzian. When restricted to \( \bigoplus_{i=1}^q a_i \), the invariant bilinear form \( \langle \ldots \rangle \) is still non-degenerate and has signature \( (q(r-1), q) \), where \( r \) is the common rank of the \( m_i \), \( i = 1, 2, \ldots, q \). Hence \( q = 1 \) and \( I_{rc} \) is connected.

**Proposition 3.26.** If \( g \) is of finite, affine or hyperbolic type, then any finite gradation is real: \( I_{im}^{I_m} = \emptyset \) and \( (I, J) \) is a \( C \)-admissible pair.

**Proof.** The result is trivial if \( g \) is of finite type. Suppose \( I_{im}^{I_m} \neq \emptyset \) for one of the other cases. If \( g \) is affine, then \( I_{rc} \) is of finite type and by Lemma 3.19 , \( m \) is contained in the finite dimensional semi-simple Lie algebra \( \mathfrak{g}(I_{rc}) \). This contradicts the fact that \( m \) is, as \( g \), of affine type (see Proposition 3.6). If \( g \) is hyperbolic, then it is Lorentzian and perfect (cf. section 1.1). By Lemma 3.20 and Corollary 3.25, \( \mathfrak{g}(I_{rc}) \) is an indecomposable finitely \( \Sigma \)-graded Kac-Moody subalgebra of \( g \). As \( I_{rc} \)
is assumed to be a proper connected subset of \( I \), \( \mathfrak{g}(I_{rc}) \) is of finite or affine type, a contradiction since, by Proposition 3.6, \( m \) must be Lorentzian. Hence \( I_{im}^{I_m} = \emptyset \) in the two last cases.

**Proposition 3.27.** If \( g \) is hyperbolic, then the grading subalgebra \( m \) is also hyperbolic.

**Proof.** Recall that in this case, \( I_{rc} = I \) (see Proposition 3.26 and Corollary 3.15). Let \( \bar{I}^1 \) be a proper subset of \( I \) and suppose that \( \bar{I}^1 \) is connected. Let \( I^1 = \bigcup_{s \in \bar{I}^1} (\bigcup_{k \in \bar{I}} I_k) \). Then, \( I^1 \) is a proper subset of \( I \). We may assume that the subalgebra \( \mathfrak{m}(\bar{I}^1) \) of \( m \) is contained in \( \mathfrak{g}(I^1) \). Let \( \Sigma^1 := \Sigma(I^1) \) be the root system
of $\mathfrak{m}(\bar{F})$. Then, it is not difficult to check that $\mathfrak{g}(I^1)$ is finitely $\Sigma^1$-graded. The argument used in Proposition 3.24 shows that the indecomposable components of $\mathfrak{g}(I^1)$ (which all are of finite or affine type) are finitely $\Sigma^1$-graded. By Proposition 3.6, $\mathfrak{m}(\bar{F})$ is of finite or affine type. Hence $\mathfrak{m}$ is hyperbolic.

**Corollary 3.28.** The problem of classification of finite real gradations of $\mathfrak{g}$ comes down first to classify the $C$-admissible pairs $(I,J)$ of $\mathfrak{g}$ and then the maximal finite gradations of the corresponding admissible algebra $\mathfrak{g}^I$. When $\mathfrak{g}$ is of finite, affine or hyperbolic type, we get thus all finite gradations.

**Proof.** This follows from Proposition 3.26, Proposition 3.21 and Lemma 1.5. □

### 4. Maximal gradations

We assume now moreover that $\mathfrak{g}$ is maximally finitely $\Sigma$-graded. We keep the notations in section 3 but we have $J = I'_m = \emptyset$. So $\mathcal{T}$ is a quotient of $I$, with quotient map $\rho$ defined by $\rho_s(\alpha_k) = \gamma_{\rho(k)}$. For $s \in \mathcal{T}$, $\Gamma_s = \rho^{-1}(\{s\})$.

**Proposition 4.1.**

1) If $k \neq l \in I$ and $\rho(k) = \rho(l)$, then there is no link between $k$ and $l$ in the Dynkin diagram of $A$: $\alpha_k(\alpha_l') = \alpha_l(\alpha_k') = 0$ and $\langle \alpha_k, \alpha_l \rangle = 0$.

2) $a \subset \{ h \in \mathfrak{h} \mid \alpha_k(h) = \alpha_l(h) \text{ whenever } \rho(k) = \rho(l) \}$.

3) For good choices of the simple coroots and Chevalley generators $(\alpha_k', e_k, f_k)_{k \in I}$ in $\mathfrak{g}$ and $(\gamma_k, X_s, Y_s)_{s \in \mathcal{T}}$ in $\mathfrak{m}$, we have $\gamma_s = \sum_{k \in \Gamma_s} \alpha_k'$, $X_s = \sum_{k \in \Gamma_s} e_k$ and $Y_s = \sum_{k \in \Gamma_s} f_k$.

4) In particular, for $s, t \in \mathcal{T}$, we have $\gamma_s(\gamma_t') = \sum_{k \in \Gamma_s} \alpha_i(\alpha_k')$ for any $i \in \Gamma_s$.

**Proof.** Assertions 1) and 2) are proved in 3.14 and 3.18. For $i \in \Gamma_s$, $\gamma_s = \rho_s(\alpha_i)$ is the restriction of $\alpha_i$ to $\mathfrak{a}$; so 4) is a consequence of 3).

For 3) recall the proof of Proposition 3.14. The $\mathfrak{sl}_2$-triple $(X_s, \gamma_s, Y_s)$ may be written $\gamma_s = \sum_{k \in \Gamma_s} H_k$, $X_s = \sum_{k \in \Gamma_s} E_k$ and $Y_s = \sum_{k \in \Gamma_s} F_k$ where $(E_k, H_k, F_k)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}(I_k)$, with $\alpha_k(H_k) = 2$. But now $J = I'_m = \emptyset$, so $I_k = \{k\}$ and $\mathfrak{g}(I_k) = C_{k} \oplus C_{\alpha_k'} \oplus C_{f_k}$, hence the result. □

So the grading subalgebra $\mathfrak{m}$ may be entirely described by the quotient map $\rho$.

We look now to the reciprocal construction.

So $\mathfrak{g}$ is an indecomposable and symmetrizable Kac-Moody algebra associated to a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$. We consider a quotient $\mathcal{T}$ of $I$ with quotient map $\rho : I \to \mathcal{T}$ and fibers $\Gamma_s = \rho^{-1}(\{s\})$ for $s \in \mathcal{T}$. We suppose that $\rho$ is an admissible quotient i.e. that it satisfies the following two conditions:

- **(MG1)** If $k \neq l \in I$ and $\rho(k) = \rho(l)$, then $a_{k,l} = \alpha_l(\alpha_k') = 0$.
- **(MG2)** If $s \neq t \in \mathcal{T}$, then $\bar{\pi}_{s,t} := \sum_{i \in \Gamma_s} a_{i,j} = \sum_{i \in \Gamma_s} \alpha_j(\alpha_i')$ is independent of the choice of $j \in \Gamma_t$.

**Proposition 4.2.** The matrix $\mathcal{A} = (\bar{\pi}_{s,t})_{s,t \in \mathcal{T}}$ is an indecomposable generalized Cartan matrix.

**Proof.** Let $s \neq t \in \mathcal{T}$ and let $j \in \Gamma_t$. By (MG1) one has $\bar{a}_{s,t} = \sum_{i \in \Gamma_s} a_{i,j} = a_{j,j} = 2$, and by (MG2) $\bar{\pi}_{s,t} := \sum_{i \in \Gamma_s} a_{i,j} \in \mathbb{Z}^\times$ (\forall j \in \Gamma_t). Moreover, $\bar{\pi}_{s,s} = 0$ if and only
if \( a_{i,j} = 0(=a_{j,i}), \forall (i,j) \in \Gamma_s \times \Gamma_t \). It follows that \( \pi_{s,t} = 0 \) if and only if \( \pi_{t,s} = 0 \), and \( A \) is a generalized Cartan matrix. Since \( A \) is indecomposable, \( \bar{A} \) is also indecomposable. \( \square \)

Let \( \mathfrak{h}^\Gamma = \{ h \in \mathfrak{h} \mid \alpha_k(h) = \alpha_l(h) \text{ whenever } \rho(k) = \rho(l) \}, \gamma_s^\vee = \sum_{k \in \Gamma_s} \alpha_k^\vee \) and \( \mathfrak{a}' = \oplus_{s \in \bar{T}} \mathbb{C}\gamma_s^\vee \subset \mathfrak{h}^\Gamma \). We may choose a subspace \( \mathfrak{a}' \) in \( \mathfrak{h}^\Gamma \) such that \( \mathfrak{a}' \cap \mathfrak{a} = \{0\} \), the restrictions \( \pi_s = : \gamma_{\rho(i)} \) to \( \mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}'' \) of the simple roots \( \alpha_i \) (corresponding to different \( \rho(i) \in \bar{T} \)) are linearly independent and \( \mathfrak{a}'' \) is minimal for these two properties.

**Proposition 4.3.** \( \{a, \{\gamma_s \mid s \in I\}, \{\gamma_s^\vee \mid s \in \bar{T}\} \) is a realization of \( \bar{A} \).

**Proof.** Let \( \ell \) be the rank of \( \bar{A} \). Note that \( \mathfrak{a} \) contains \( \mathfrak{a}' = \oplus_{s \in \bar{T}} \mathbb{C}\gamma_s^\vee \); the family \( \langle \gamma_s \rangle_{s \in I} \) is free in the dual space \( \mathfrak{a}^* \) of \( \mathfrak{a} \) and satisfies \( \langle \gamma_t, \gamma_s^\vee \rangle = \bar{a}_{s,t}, \forall s, t \in \bar{I} \). It follows that \( \dim(\mathfrak{a}) \geq 2|\bar{I}| - \ell \) (see [11, 14.1] or [12, Ex. 1.3]). As \( \mathfrak{a} \) is minimal, we have \( \dim(\mathfrak{a}) = 2|\bar{I}| - \ell \) (see [11, 14.2] for minimal realization). Hence \( \{a, \{\gamma_s \mid s \in I\}, \{\gamma_s^\vee \mid s \in \bar{T}\} \) is a (minimal) realization of \( \bar{A} \). \( \square \)

We note \( \Delta^\rho = \Sigma \oplus_{s \in \bar{I}} 2\gamma_s \) the root system associated to this realization.

We define now \( \mathfrak{X}_s = \sum_{k \in \Gamma_s} e_k \) and \( \mathfrak{Y}_s = \sum_{k \in \Gamma_s} f_k \). Let \( \mathfrak{m} = \mathfrak{g}^\rho \) be the Lie subalgebra of \( \mathfrak{g} \) generated by \( \mathfrak{a} \) and the elements \( \mathfrak{X}_s, \mathfrak{Y}_s \) for \( s \in \bar{T} \).

**Proposition 4.4.** The Lie subalgebra \( \mathfrak{m} = \mathfrak{g}^\rho \) is the Kac-Moody algebra associated to the realization \( \{a, \{\gamma_s \mid s \in I\}, \{\gamma_s^\vee \mid s \in \bar{T}\} \) of \( \bar{A} \). Moreover, \( \mathfrak{g} \) is an integrable \( \mathfrak{g}^\rho - \text{module with finite multiplicities.} \)

**Proof.** Clearly, the following relations hold in the Lie subalgebra \( \mathfrak{g}^\rho \):

\[
\begin{align*}
[a, a] &= 0, \\
[a, \mathfrak{X}_s] &= \langle \gamma_s, a \rangle \mathfrak{X}_s, \\
[a, \mathfrak{Y}_s] &= \delta_{s,t} \gamma_s^\vee (s, t \in \bar{I}); \\
[\mathfrak{X}_s, \mathfrak{Y}_t] &= \delta_{s,t} \gamma_s^\vee (s, t \in I).
\end{align*}
\]

For the Serre's relations, one has :

\[1 - \pi_{s,t} \geq 1 - a_{i,j}, \forall (i, j) \in \Gamma_s \times \Gamma_t.\]

In particular, one can see, by induction on \( |\Gamma_s| \), that :

\[(\text{ad} \mathfrak{X}_s)^{1 - \pi_{s.t}} (e_j) = (\sum_{i \in \Gamma_s} \text{ad} e_i)^{1 - \pi_{s.t}} (e_j) = 0, \forall j \in \Gamma_t.\]

Hence

\[(\text{ad} \mathfrak{X}_s)^{1 - \pi_{s.t}} (\mathfrak{Y}_t) = 0, \forall s, t \in \bar{I}.\]

and in the same way we obtain that :

\[(\text{ad} \mathfrak{Y}_s)^{1 - \pi_{s.t}} (\mathfrak{Y}_t) = 0, \forall s, t \in I.\]

It follows that \( \mathfrak{g}^\rho \) is a quotient of the Kac-Moody algebra \( \mathfrak{g}(\bar{A}) \) associated to \( \bar{A} \) and \( \{a, \{\gamma_s \mid s \in \bar{T}\}, \{\gamma_s^\vee \mid s \in \bar{T}\} \) in which the Cartan subalgebra \( \mathfrak{a} \) of \( \mathfrak{g}(\bar{A}) \) is embedded. By [12, 1.7] \( \mathfrak{g}^\rho \) is equal to \( \mathfrak{g}(\bar{A}) \).

It's clear that \( \mathfrak{g} \) is an integrable \( \mathfrak{g}^\rho \)-module with finite dimensional weight spaces relative to the adjoint action of \( \mathfrak{a} \), since for \( \alpha = \sum_{i \in \bar{I}} n_i \alpha_i \in \Delta^\rho \), its restriction \( \rho_a(\alpha) \) to \( \mathfrak{a} \) is given by

\[
\rho_a(\alpha) = \sum_{s \in \bar{I}} (\sum_{i \in \Gamma_s} n_i) \gamma_s
\]
Proposition 4.5. The Kac-Moody algebra $g$ is maximally finitely $\Delta^\rho$–graded with grading subalgebra $g^\rho$.

Proof. As in Theorem 2.14, we will see that $\rho_a(\Delta^+) \subset Q_+^\Gamma := \bigoplus_{s \in \bar{I}} \mathbb{Z}^+ \gamma_s$ satisfies, as $\Sigma^+ = \Delta^\rho_\Delta$, the following conditions:

(i) $\gamma_s \in \rho_a(\Delta^+) \subset Q_+^\Gamma$, $2\gamma_s \notin \rho_a(\Delta^+)$, $\forall s \in \bar{I}$.
(ii) if $\gamma \in \rho_a(\Delta^+)$, $\gamma \neq \gamma_s$, then the set $\{\gamma + k\gamma_s; k \in \mathbb{Z}\} \cap \rho_a(\Delta^+)$ is a string $\{\gamma - p\gamma_s, ..., \gamma + q\gamma_s\}$, where $p, q \in \mathbb{Z}^+$ and $p - q = \langle \gamma, \gamma_s \rangle$;
(iii) if $\gamma \in \rho_a(\Delta^+)$, then $\text{supp}(\gamma)$ is connected.

Clearly $\{\gamma_s \mid s \in \bar{I}\} \subset \rho_a(\Delta_+) \subset Q_+^\Gamma$. For $\alpha \in \Delta$ and $s \in \bar{I}$, the condition $\rho_a(\alpha) \in \mathbb{N}\gamma_s$ implies $\alpha \in \Delta(\Gamma_s)^+ = \{\alpha_i; i \in \Gamma_s\}$ [see (4.1)]. It follows that $2\gamma_s \notin \rho_a(\Delta^+)$ and (i) is satisfied. By Proposition 4.4, $g$ is an integrable $g^\rho$–module with finite multiplicities. Hence, the property (ii) follows from [12, 3.6]. Let $\alpha \in \Delta_+$ and let $s, t \in \text{supp}(\rho_a(\alpha))$. By (4.1) there exists $(k, l) \in \Gamma_s \times \Gamma_t$ such that $k, l \in \text{supp}(\alpha)$, which is connected. Hence there exist $i_0 = k, i_1, ..., i_{n+1} = l$ such that $\alpha_{i_j} \in \text{supp}(\alpha), j = 0, 1, ..., n+1$, and for $j = 0, 1, ..., n, i_j$ and $i_{j+1}$ are linked relative to the generalized Cartan matrix $A$. In particular, $\rho(i_j) \neq \rho(i_{j+1}) \in \text{supp}(\rho_a(\alpha))$ and they are linked relative to the generalized Cartan matrix $\bar{A}, j = 0, 1, ..., n$, with $\rho(i_0) = s$ and $\rho(i_{n+1}) = t$. Hence the connectedness of $\text{supp}(\rho_a(\alpha))$ relative to $\bar{A}$. It follows that $\rho_a(\Delta^+) = \Delta^\rho_\Delta$ and hence $\rho_a(\Delta) = \Delta^\rho$ (see [12, Ex. 5.4]). In particular, $g$ is finitely $\Delta^\rho$–graded with $J = \emptyset = I_{\text{ad}}'$.

\[\Box\]

Corollary 4.6. The restriction to $m = g^\rho$ of the invariant bilinear form $(\cdot, \cdot)$ of $g$ is non-degenerate. In particular, the generalized Cartan matrix $\bar{A}$ is symmetrizable of the same type as $A$.

Proof. The first part of the corollary follows from Proposition 4.5 and Corollary 3.17. The second part follows from Proposition 3.6.

\[\Box\]

Remark 4.7. The map $\rho$ coincides with the map (also denoted $\rho$) defined at the beginning of this section using the maximal gradation of Proposition 4.5. Conversely Proposition 4.1 tells that, for a general maximal finite gradation, $\rho$ is admissible and $m = g^\rho$ for good choices of the Chevalley generators. So we get a good correspondence between maximal gradations and admissible quotient maps.

By Corollary 3.28 the real finite gradations of a Kac-Moody algebra $g$ are bijectively associated to pairs of a $C$–admissible pair $(I, J)$ and an admissible quotient map $\rho : I' = I \setminus J \to \bar{I}'$.

5. An example

The following example shows that imaginary gradations do exist. It shows in particular that, for a generalized $C$–admissible pair $(I, J)$, $J^0$ may be non-empty and $I_{\text{re}}$ may be non-connected. Moreover, the Kac-Moody algebra $g$ may be not graded by the root system of $\bar{g}(I_{\text{re}})$.

The imaginary gradations will be studied in a forthcoming paper [7].
Example 5.1. Consider the Kac Moody algebra $\mathfrak{g}$ corresponding to the indecomposable and symmetric generalized Cartan matrix $A$:

$$
A = \begin{pmatrix}
2 & -3 & -1 & 0 & 0 & 0 \\
-3 & 2 & -1 & 0 & 0 & 0 \\
-1 & -1 & 2 & -1 & -1 & -1 \\
0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & -1 & 0 & 2 & -3 \\
0 & 0 & -1 & 0 & -3 & 2
\end{pmatrix}
$$

with the corresponding Dynkin diagram:

```
   3 \_ 3
3 \_ 2 \_ 3
   2 \_ 4
```

Note that $\det(A) = 275$ and the symmetric submatrix of $A$ indexed by $\{1, 2, 4, 5, 6\}$ has signature $(+++-,-)$. Since $\det(A) > 0$, the matrix $A$ should have signature $(+++-,--)$. Let $\Sigma$ be the root system associated to the strictly hyperbolic generalized Cartan matrix $\left(\begin{array}{cc}2 & -3 \\ -3 & 2\end{array}\right)$, the corresponding Dynkin diagram is the following:

```
H_{3,3}
```

We will see that $\mathfrak{g}$ is finitely $\Sigma$-graded and describe the corresponding generalized $C$-admissible pair.

1) Let $\tau$ be the involutive diagram automorphism of $\mathfrak{g}$ such that $\tau(1) = 5$, $\tau(2) = 6$ and $\tau$ fixes the other vertices. Let $\sigma'_n$ be the normal semi-involution of $\mathfrak{g}$ corresponding to the split real form of $\mathfrak{g}$. Consider the quasi-split real form $\mathfrak{g}_R^1$ associated to the semi-involution $\tau \sigma'_n$ (see [2] or [6]). Then $t_R := h_R^1$ is a maximal split toral subalgebra of $\mathfrak{g}_R^1$. The corresponding restricted root system $\Delta' := \Delta(\mathfrak{g}_R, t_R)$ is reduced and the corresponding generalized Cartan matrix $A'$ is given by:

$$
A' = \begin{pmatrix}
2 & -3 & -2 & 0 \\
-3 & 2 & -2 & 0 \\
-1 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}
$$

with the corresponding Dynkin diagram:

```
   3 \_ 3
3 \_ 2 \_ 3
   2 \_ 4
```

Following N. Bardy [4, 9], there exists a split real Kac-Moody subalgebra $m_R^1$ of $\mathfrak{g}_R^1$ containing $t_R$ such that $\Delta' = \Delta(m_R^1, t_R)$. It follows that $\mathfrak{g}$ is finitely $\Delta'$-graded.

2) Let $m^1 := m_R^1 \otimes \mathbb{C}$ and $t := t_R \otimes \mathbb{C}$. Denote by $\alpha'_i := \alpha_i/t$, $i = 1, 2, 3, 4$. Put $\alpha'_1 = \alpha_1 + \alpha_5$, $\alpha'_2 = \alpha_2 + \alpha_6$, $\alpha'_3 = \alpha_3$ and $\alpha'_4 = \alpha_4$. Let $I^1 := \{1, 2, 3, 4\}$, then $(t, \Pi' = \{\alpha'_i, \; i \in I^1\}, \Pi'^\vee = \{\alpha'^*_i, \; i \in I^1\})$ is a realization of $A'$ which is symmetrizable and Lorentzian.

Let $m$ be the Kac-Moody subalgebra of $m^1$ corresponding to the submatrix $\bar{A}$ of $A'$ indexed by $\{1, 2\}$. Thus $\bar{A} = \left(\begin{array}{rr}2 & -3 \\ -3 & 2\end{array}\right)$ is strictly hyperbolic. Let $a :=$
\( \mathfrak{a}'_1 \oplus \mathfrak{a}'_2 \) be the standard Cartan subalgebra of \( \mathfrak{m} \) and let \( \Sigma = \Delta(\mathfrak{m}, \mathfrak{a}) \). For \( \alpha' \in \mathfrak{t}^* \), denote by \( \rho_1(\alpha') \) the restriction of \( \alpha' \) to \( \mathfrak{a} \). Put \( \gamma_s = \rho_1(\alpha'_s) \), \( \gamma_s' = \alpha'_s \), \( s = 1, 2 \). Then \( \Pi_0 = \{ \gamma_1, \gamma_2 \} \) is the standard root basis of \( \Sigma \). One can see easily that \( \rho_1(\alpha'_1) = 0 \) and \( \rho_1(\alpha'_2) = 2(\gamma_1 + \gamma_2) \) is a strictly positive imaginary root of \( \Sigma \). Now we will show that \( \mathfrak{m}^1 \) is finitely \( \Sigma \)-graded.

Let \((\ldots,)_1 \) be the normalized invariant bilinear form on \( \mathfrak{m}^1 \) such that short real roots have length 1 and long real roots have square length 2. Then there exists a positive rational \( q \) such that the restriction of \((\ldots,)_1 \) to \( \mathfrak{t} \) has the matrix \( B_1 \) in the basis \( \Pi_0^* \), where:

\[
B_1 = q^{-1} \begin{pmatrix}
2 & -3 & -1 & 0 \\
-3 & 2 & -1 & 0 \\
-1 & -1 & 1 & -1/2 \\
0 & 0 & -1/2 & 1
\end{pmatrix}
\]

By duality, the restriction of \((\ldots,)_1 \) to \( \mathfrak{a} \) induces a non-degenerate symmetric bilinear form on \( \mathfrak{t}^* \) (see [12, 2.1]) such that its matrix \( B_1^\dagger \) in the basis \( \Pi_0^* \), is the following:

\[
B_1^\dagger = q^{-1} \begin{pmatrix}
2 & -3 & -2 & 0 \\
-3 & 2 & -2 & 0 \\
-2 & -2 & 4 & -2 \\
0 & 0 & -2 & 4
\end{pmatrix}
\]

Hence, \( q \) equals 2.

Note that for \( \alpha' = \sum_{i=1}^4 n_i \alpha'_i \in \Delta'^+ \), we have that
\[
(\alpha', \alpha')_1 = n_1^2 + n_2^2 + 2n_3^2 + 2n_4^2 - 3n_1n_2 - 2n_1n_3 - 2n_2n_3 - 2n_3n_4.
\]

We will show that \( \rho_1(\Delta'^+) = \Sigma^+ \cup \{0\} \). Note that \( \Sigma \) can be identified with \( \Delta' \cap (\mathbb{Z} \alpha'_1 + \mathbb{Z} \alpha'_2) \); hence \( \rho_1 \) is injective on \( \Sigma \) and \( \Sigma^+ \subset \rho_1(\Delta'^+) \).

Let \((\ldots,)_a \) be the normalized invariant bilinear form on \( \mathfrak{m} \) such that all real roots have length 2. Then the restriction of \((\ldots,)_a \) to \( \mathfrak{a} \) has the matrix \( B_a \) in the basis \( \Pi_a = \{ \gamma_1, \gamma_2 \} \), where:

\[
B_a = \begin{pmatrix}
2 & -3 & -3 \n-3 & 2 & 2 \n0 & 0 & 4
\end{pmatrix}
\]

Since \( A \) is symmetric, the non-degenerate symmetric bilinear form, on \( \mathfrak{a}^* \), induced by the restriction of \((\ldots,)_a \) to \( \mathfrak{a} \), has the same matrix \( B_a \) in the basis \( \Pi_a \). In particular, we have that:

\[
(p_1(\alpha'), p_1(\alpha')_a) = 2[(n_1 + 2n_3)^2 + (n_2 + 2n_3)^2 - 3(n_1 + 2n_3)(n_2 + 2n_3)],
\]

since \( p_1(\alpha') = (n_1 + 2n_3)\gamma_1 + (n_1 + 2n_3)\gamma_2 \).

Using (5.1), it is not difficult to check that
\[
(p_1(\alpha'), p_1(\alpha')_a) = 2[(\alpha', \alpha')_1 - (n_3 - n_4)^2 - 5n_3^2 - n_4^2]
\]

Suppose \( n_3 = 0 \), then, since \( \text{supp}(\alpha') \) is connected, we have that \( \alpha' = n_1 \alpha'_1 + n_2 \alpha'_2 \), or \( \alpha' = \alpha'_4 \). Hence \( p_1(\alpha') = n_1 \gamma_1 + n_2 \gamma_2 \in \Sigma \) or \( p_1(\alpha') = 0 \).

Suppose \( n_3 \neq 0 \), then, since \( (\alpha', \alpha')_1 \leq 2 \), one can see, using (5.2), that

\[
(p_1(\alpha'), p_1(\alpha'))_a < 0.
\]

As \( \Sigma \) is hyperbolic and \( p_1(\alpha') \in \mathbb{N} \gamma_1 + \mathbb{N} \gamma_2 \), we deduce that \( p_1(\alpha') \) is a positive imaginary root of \( \Sigma \) (see [12, 5.10]). It follows that \( \rho_1(\Delta'^+) = \Sigma^+ \cup \{0\} \).

To see that \( \mathfrak{m}^1 \) is finitely \( \Sigma \)-graded, it suffices to prove that, for \( \gamma = m_1 \gamma_1 + m_2 \gamma_2 \in \Sigma^+ \cup \{0\} \), the set \( \{ \alpha' \in \Delta'^+, p_1(\alpha') = \gamma \} \) is finite. Note that if \( \alpha' = \sum_{i=1}^4 n_i \alpha'_i \in \Delta'^+ \) satisfying \( p_1(\alpha') = \gamma \), then \( n_i + 2n_3 = m_i \), \( i = 1, 2 \). In particular, there are
only finitely many possibilities for \( n_i \), \( i = 1, 2, 3 \). The same argument as the one used in the proof of Proposition 2.13 shows also that there are only finitely many possibilities for \( n_4 \).

3) Recall that \( m \subset m^1 \subset g \). The fact that \( g \) is finitely \( \Delta' \)–graded with grading subalgebra \( m^0 \) and \( m^1 \) is finitely \( \Sigma \)–graded implies that \( g \) is finitely \( \Sigma \)–graded (cf. lemma 1.5). Let \( I = \{1, 2, 3, 4, 5, 6\} \), then the root basis \( \Pi_o \) of \( \Sigma \) is adapted to the root basis \( \Pi \) of \( \Delta \) and we have \( I_{re} = \{1, 2, 5, 6\} \) (not connected), \( \Gamma_1 = \{1, 5\} \), \( \Gamma_2 = \{2, 6\} \), \( J = \{4\} \), \( J_{re} = \emptyset \), \( I'_{re} = \{3\} \) and \( J^o = J = \{4\} \).

Note that, for this example, \( g(I_{re}) \), which is \( \Sigma \)–graded, is isomorphic to \( m \times m \). This gradation corresponds to that of the pseudo-complex real form of \( m \times m \) (i.e. the complex Kac-Moody algebra \( m \) viewed as real Lie algebra) by its restricted reduced root system. Since the pair \((I_3, J_3) = (\{3, 4\}, \{4\})\) is not admissible, it is not possible to build a Kac-Moody algebra \( g \) grading finitely \( g \) and maximally finitely \( \Sigma \)–graded.

Acknowledgments We thank the anonymous referee for his/her valuable comments and suggestions.

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