A remark on Sarnak’s conjecture

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Abstract. We investigate Sarnak’s conjecture on the Möbius function in the special case when the test function is the indicator of a level set of an additive function with restricted values at primes.

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1. Introduction and statements of results

According to a general pseudo-randomness principle related to a famous conjecture of Chowla [1] and recently considered by Sarnak [7], the Möbius function \( \mu \) does not correlate with any function \( \xi \) of low complexity. In other words,

\[
\sum_{n \leq x} \mu(n) \xi(n) = o \left( \sum_{n \leq x} |\xi(n)| \right) \quad (x \to \infty).
\]

There are many ways of constructing functions of low complexity. Sarnak and others use return times of sampling sequences of a dynamical system, which leads to a natural measure of the complexity. Here we propose to follow another path by selecting the test-function as the indicator of a level set of a real additive function \( f \). Since computing the value of such indicator at an integer \( n \) rests on the factorization of \( n \), it is certainly not deterministic in the sense introduced by Sarnak. However, as we shall see, this setting provides a continuous measure of correlations as \( f \) departs from a constant multiple of the total number of prime factors function.

It is known since Halász [5] that, for any additive function \( f \), we have

\[
Q(x; f) := \sup_{m \in \mathbb{R}} \sum_{n \leq x, f(n) = m} 1 \ll \frac{x}{\sqrt{1 + E(x)}}
\]

where we have put

\[
E(x) := \sum_{p \leq x, f(p) \neq 0} \frac{1}{p}.
\]

Here and in the sequel, the letter \( p \) denotes a prime number.

The estimate (1.2) is known to be optimal in this generality since the two sides achieve the same order of magnitude when \( f(n) \) is equal to the total number of prime factors of \( n \), counted with or without multiplicity.

As a first investigation of the above described problem, we would like to show that

\[
Q(x; f, \mu) := \sup_{m \in \mathbb{R}} \left| \sum_{n \leq x, f(n) = m} \mu(n) \right|
\]
is generically smaller than the right-hand side of (1·2). Of course we have to avoid the case when \( f(p) \) is constant, for then \( \mu(n) \) does not oscillate on the set of squarefree integers \( n \) with \( f(n) = m \). Therefore we seek an estimate which matches (1·2) when \( f(p) \) is close to a constant and has smaller order of magnitude otherwise.

When \( f(p) \) is restricted to assume the values 0 or 1 only, we thus expect a significant improvement over (1·2) when (1·3)

\[
F(x) := \sum_{p \leq x} \frac{1 - f(p)}{p}
\]

is large. Indeed, in this simple case we obtain the following estimate.

**Theorem 1.1.** Let \( f \) denote a real additive arithmetic function such that \( f(p) \in \{0, 1\} \) for all \( p \). Then, with the above notation and \( c = (2\pi - 4)/(3\pi - 2) \approx 0.30751 \), we have (1·4)

\[
Q(x; f, \mu) \ll \frac{x\{1 + F(x)\}e^{-cF(x)}}{\sqrt{1 + E(x)}}.
\]

For simplicity, let us retain in the sequel the hypothesis \( f(p) \in \{0, 1\} \). Under the assumption that \( F(x) \), as defined in (1·3) above, grows sufficiently slowly, we may prove an estimate that is valid for each \( m \) in a large range around the mean, and so may be stated in the exact frame of Sarnak’s conjecture.

Let us denote by \( N_m(x; f) \) the number of squarefree integers not exceeding \( x \) such that \( f(n) = m \). It follows from results of Halász [3, 4], and Sárközy [6] that, given any \( \kappa \in ]0, 1[ \), we have (1·5)

\[
N_m(x; f) \asymp x \frac{E(x)^m}{m!} e^{-E(x)} \quad (\kappa E(x) \leq m \leq E(x)/\kappa).
\]

Moreover, Halász announced (see [2], p. 312) the possibility to obtain, in the same range for \( m \), an asymptotic formula for \( N_m(x; f) \), a result which actually follows, as shown in [10], from a general effective mean value estimate for multiplicative functions established in the same work—see below.

This supports the hope to obtain an asymptotic formula for

\[
N_m(x; f, \mu) := \sum_{n \leq x \atop f(n) = m} \mu(n)
\]

which directly compares to (1·5). In view of (1·1), we may assume with no loss of generality that \( f \) is strongly additive. We obtain the following result. Here and in the sequel we let \( \log_k \) denote the \( k \)-fold iterated logarithm.

**Theorem 1.2.** Let \( \kappa \in ]0, 1[ \) and let \( f \) denote a strongly additive function such that \( f(p) \in \{0, 1\} \) for all primes \( p \). Assume furthermore that

(1·6)

\[
F(x) := \sum_{p \leq x} \frac{1 - f(p)}{p} \ll \log_3 x \quad (x \to \infty)
\]

(1·7)

\[
\sum_{\text{exp}\{(\log x)/(\log_2 x)\} < p \leq y} \frac{(1 - f(p)) \log p}{p} \ll \frac{\log y}{(\log_2 x)^c_0} \quad (x^{1/(\log_2 x)^c} < y \leq x)
\]

\[1. \text{ All our results could be straightforwardly adapted to the case when } f(p) \text{ is restricted to a fixed, finite set, or even to a set of moderate size depending on } x.\]
where $D$ and $c_0$ are positive constants. Provided $D$ is sufficiently large and uniformly in the range $\kappa E(x) \leq m \leq E(x)/\kappa$, we have

$$N_m(x; f, \mu) = (-1)^m N_m(x; f) \left\{ \lambda_f e^{-2F(x)} + O\left( \frac{1}{(\log x)^b} \right) \right\},$$

with

$$\lambda_f := \prod_{f(p) = 0} \frac{1 - 1/p}{1 + 1/p} \exp^{2p}, \quad b := \frac{1}{2} \min\{1, c_0\kappa/(4 - \kappa)\}.$$

To fix ideas, note that a strongly additive function $f$ such that $f(p) \in \{0, 1\}$ satisfies hypotheses (1-6) and (1-7) as soon as

$$\sum_{p \leq y} (1 - f(p)) \log p \ll \frac{y}{(\log y)^{\max\{1, c_0\}}}.$$

The proof of Theorem 1.2 rests on the following recent result of the second author [10] (Theorem 1.4), for the statement of which we introduce further notation. We let $M(A, B)$ designate the class of those complex-valued multiplicative functions $g$ such that

$$\max_p |g(p)| \leq A, \quad \sum_{p, \nu \geq 2} \frac{|g(p^\nu)| \log p^\nu}{p^\nu} \leq B,$$

and, for $b \in \mathbb{R}$, we write

$$\beta_0 = \beta_0(b, A) := 1 - \frac{\sin(2\pi b/A)}{2\pi b/A}.$$

Moreover, given a complex-valued function $g$, we put $w_g := 1$ if $g$ is real, $w_g := \frac{1}{2}$ otherwise, and write

$$M(x; g) := \sum_{n \leq x} g(n), \quad Z(x; g) := \sum_{p \leq x} \frac{|g(p)|}{p}.$$

**Theorem 1.3 ([10]).** Let

$$a \in [0, \frac{1}{2}], \quad b \in [a, \frac{1}{2}], \quad b := (1 - b)/b, \quad A \geq 2b, \quad B > 0, \quad \beta := \beta_0(b, A),$$

$$x \geq 2, \quad 1/\sqrt{\log x} \leq \varepsilon \leq \frac{1}{2},$$

and let the multiplicative functions $g, r$, such that $r \in M(x; 2A, B), \ |g| \leq r$, satisfy the conditions

$$\sum_{p \leq x} \frac{r(p) - \Re g(p)}{p} \leq \frac{1}{2} \beta b \log(1/\varepsilon),$$

$$\sum_{x^\varepsilon < p \leq y} \frac{\{r(p) - \Re g(p)\} b \log p}{p} \ll \varepsilon \beta b \log y \quad (x^\varepsilon < y \leq x),$$

with $\delta \in [0, \frac{1}{3}\beta b]$, and

$$\min_{x^\varepsilon < p \leq x} r(p) \geq 4b.$$

We then have

$$M(x; g) = M(x; r) \prod_p \frac{\sum_{p^\nu \leq x} g(p^\nu)/p^\nu}{\sum_{p^\nu \leq x} r(p^\nu)/p^\nu} + O\left( \frac{x \varepsilon w_\delta \varepsilon \Re g(x) - \varepsilon \Re g(x; |g| - g)}{\log x} \right)$$

where $c := b/A$. The implicit constant in (1-15) depends at most upon $A$, $B$, $a$, and $b$. 


2. Proof of Theorem 1.1

Put

\[ M(x; \vartheta) := \sum_{n \leq x} \mu(n) e^{2\pi i \vartheta f(n)}. \]

Since \( f \) assumes only integer values, we plainly have, for any \( m \in \mathbb{Z} \),

\[ \sum_{n \leq x} \mu(n) = \sum_{n \leq x} \mu(n) \int_{-1/2}^{1/2} e^{2\pi i (f(n)-m) \vartheta} d\vartheta = \int_{-1/2}^{1/2} M(x; \vartheta) e^{-2\pi i m \vartheta} d\vartheta, \]

so

\[ Q(x; f, \mu) \leq \int_{-1/2}^{1/2} |M(x; \vartheta)| d\vartheta. \]

From Corollary III.4.12 in [8], we infer that, uniformly for \( \vartheta \in \mathbb{R}, T \geq 1, x \geq 1, \)

\[ (2 \cdot 1) \quad M(x; \vartheta) \leq \frac{x \{1 + m(x; \vartheta, T)\}}{e^{m(x; \vartheta, T)}} + \frac{x}{T}, \]

where we have put

\[ m(x; \vartheta, T) := \min_{|\tau| \leq T} \sum_{p \leq x} \frac{1 + \cos(2\pi \vartheta f(p) - \tau \log p)}{p}. \]

We select \( T := \log x \), so that the second term on the right of \( (2 \cdot 1) \) is negligible compared to the upper bound in \( (1 \cdot 4) \). Let \( h_\vartheta \) be defined by

\[ h_\vartheta(t) := 1 + \min\{ \cos(t), \cos(2\pi \vartheta - t) \} \quad (t \in \mathbb{R}), \]

so that

\[ s_\vartheta := \frac{1}{2\pi} \int_{-\pi}^{\pi} h_\vartheta(t) dt = 1 - \frac{2}{\pi} \left| \sin(\pi \vartheta) \right| \quad (\vartheta \in [-\frac{1}{2}, \frac{1}{2}]), \]

and, for suitable \( \tau \in [-T, T] \),

\[ (2 \cdot 2) \quad m(x; \vartheta, T) \geq \sum_{p \leq x} \frac{h_\vartheta(\tau \log p)}{p}. \]

We note that the constant \( c \) appearing in the statement of Theorem 1.1 coincides with the minimal value of \( 2s_\vartheta/(2 + s_\vartheta) \) as \( \vartheta \) ranges over \([-\frac{1}{2}, \frac{1}{2}]\).

The right-hand side of \( (2 \cdot 2) \) may be estimated via partial summation as made explicit in lemma III.4.13 of [8]. For any \( w \in [2, x] \) and \( \vartheta \in [-\frac{1}{2}, \frac{1}{2}] \), we have

\[ (2 \cdot 3) \quad \sum_{w < p \leq x} \frac{h_\vartheta(\tau \log p)}{p} = s_\vartheta \log \left( \frac{\log x}{\log w} \right) + O\left( \frac{1}{|\tau| \log w} + \frac{1 + |\tau|}{\sqrt{\log w}} \right). \]

If \( 1 \leq |\tau| \leq T \), we select \( w := \exp(\log_2 x)^2 \) to obtain

\[ m(x; \vartheta, T) \geq s_\vartheta \log_2 x + O(\log_3 x). \]

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2. Note, however, that Halász [5] gave a simple argument to show that, without loss of generality, one can assume \( f \) is integer-valued for the computation of \( Q(x; f) \). In the case of \( Q(x; f, \mu) \) a slight modification of his construction is needed to ensure that changing the range of \( f \) does not create new coincidences. This is not needed in the present context but could be of use when exploring more general situations.
Next, set
\[
\log v := (\log x) \exp \left\{ -\frac{2 \cos^2(\pi \vartheta) E(x) + 2F(x)}{2 + s_\vartheta} \right\}.
\]
If \(1/\log v < |\tau| \leq 1\), we put \(w := v\) in (2.3) and get
\[
\sum_{v < p \leq x} \frac{h_\vartheta(\tau \log p)}{p} \geq 2s_\vartheta \cos^2(\pi \vartheta) E(x) + \frac{2s_\vartheta}{2 + s_\vartheta} F(x) + O(1).
\]
And finally, if \(|\tau| \leq 1/\log v\), we trivially have
\[
\sum_{p \leq v} \frac{1 + \cos(2\pi f(p) - \tau \log p)}{p} = \sum_{p \leq v} \frac{1 + \cos(2\pi f(p))}{p} + O(1)
\leq (1 + \cos(2\pi)) \sum_{p \leq v} \frac{1}{p} + 2 \sum_{p \leq v, f(p) = 1} \frac{1}{p} + O(1)
\geq 2 \cos^2(\pi \vartheta) E(x) + 2F(x) - 2 \log \left( \frac{\log x}{\log v} \right) + O(1)
\geq \frac{2s_\vartheta \cos^2(\pi \vartheta)}{2 + s_\vartheta} E(x) + \frac{2s_\vartheta}{2 + s_\vartheta} F(x) + O(1).
\]
Therefore, we get in all cases
\[
(2.4)
m(x; \vartheta, T) \geq \frac{2s_\vartheta \cos^2(\pi \vartheta)}{2 + s_\vartheta} E(x) + \frac{2s_\vartheta}{2 + s_\vartheta} F(x) + O(1)
\geq c \cos^2(\pi \vartheta) E(x) + cF(x) + O(1).
\]
Integrating over \(\vartheta\) immediately yields the result stated.
\[\square\]

3. Proof of Theorem 1.2

Let us introduce the multiplicative function \(g(n) := \mu(n)z^{f(n)}\) with \(z := -\varrho e^{2\pi i \vartheta}, |\vartheta| \leq \frac{1}{2}, \kappa \leq \varrho \leq 1/\kappa\). Put \(r(n) := \mu(n)^2 g^{f(n)}\). From (2.4), we see that, with \(c\) as in the statement of Theorem 1.1,
\[
\sum_{p \leq x} \frac{r(p) - \Re g(p)/p^{\tau}}{p} \geq cg \sin^2(\pi \vartheta) E(x) + cgF(x) + O(1) \quad (|\tau| \leq T := \log x).
\]
We may therefore apply Corollary 2.1 of [10] to get
\[
(3.1)\quad M(x; g) \ll M(x; r) \left\{ e^{-cgE(x) \sin^2(\pi \vartheta) - cgF(x) \log x + \frac{1}{(\log x)^{\kappa}}} \right\}.
\]
With the aim of applying Cauchy’s formula to detect \(N_m(x; f, \mu)\), we next seek an estimate for \(M(x; g)\) when \(\vartheta\) is small, namely
\[
|\vartheta| \leq \vartheta_0 := K \sqrt{\frac{\log_3 x}{\log_2 x}},
\]
where \(K\) is a large constant—actually any \(K > 1/\sqrt{4\kappa c}\) will do. We have
\[
\sum_{p \leq x} \frac{r(p) - \Re g(p)}{p} = \varrho(1 - \cos 2\pi \vartheta) E(x) + 2\varrho F(x) \leq 2\varrho \pi^2 \vartheta^2 + 2\varrho F(x),
\]
hence condition (1.12) is plainly fulfilled with $\varepsilon := |\vartheta|^{2/\delta} + (\log_2 x)^{-c_0/(b\delta)}$ provided $\delta$ is chosen sufficiently small in terms of $b$, $\kappa$ and $K$. Next, for $x^\varepsilon < y \leq x$, we have

$$\sum_{x^\varepsilon < p \leq y} \frac{(r(p) - \Re g(p))^b \log p}{p} \leq g\vartheta^{2b} \log y + \varepsilon \sum_{x^\varepsilon < p \leq y} \frac{\log p}{p}$$

$$\leq \kappa \left\{ \varepsilon^b + (\log_2 x)^{-c_0} \right\} \log y,$$

so hypothesis (1.13) is also verified. Since (1.14) holds trivially on selecting $b := \kappa/4$, and hence $b = 4/\kappa - 1$, we conclude that (1.15) is valid. We obtain, with $c := \kappa b$,

$$M(x; g) = M(x; r) \prod_{p \leq x, f(p) = 1} \frac{1 - z/p}{1 + \varrho/p} \prod_{p \leq x, f(p) = 0} \frac{1 - 1/p}{1 + 1/p} + O \left( \frac{x^{\varepsilon^2/2} e^{Z(x;r) - \varepsilon Z(x;r-g)}}{\log x} \right).$$

Now, appealing for instance to theorem 1.1 of [9], we observe that

$$M(x; r) > \frac{xe^{Z(x;r)}}{\log x}$$

and so we may rewrite (3.2) as

$$M(x; g) = M(x; r) \left\{ \lambda e^{-(z+\vartheta)E(x) - 2F(x)} + O \left( \left| \vartheta \right| + (\log_2 x)^{-c_0} e^{c_1 |\vartheta|^2 E(x) - c_1 F(x)} \right) \right\},$$

valid for $|\vartheta| \leq \vartheta_0$ and some constant $c_1 > 0$. Integrating on the circle $|z| = \varrho := m/E(x)$ and taking (3.1) into account, we readily obtain, in the stated range for $m$,

$$N_m(x; f, \mu) = (-1)^m \int_{-1/2}^{1/2} e^{-2i\pi \vartheta m} e^{-m M(x; g)} d\vartheta$$

$$= (-1)^m \lambda M(x; r) \frac{E(x)^m}{m! e^m} \left\{ e^{-2F(x)} + O \left( \frac{e^{-c_2 F(x)}}{(\log_2 x)^b} \right) \right\},$$

with $c_2 := \min(c_1, c\kappa)$. Since, by a straightforward variant of corollary 2.4 of [10] ,

$$N_m(x; f) = M(x; r) \frac{E(x)^m}{m! e^m} \left\{ 1 + O \left( \frac{1}{\sqrt{\log_2 x}} \right) \right\},$$

we reach the required conclusion.

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References


3. Applied to $\omega(n; E)$ instead of $\Omega(n; E)$, with the notation of [10].
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