Generalized Mertens sums

Gérard Tenenbaum

To Krishna Alladi, half-way,
as a token of a life-long friendship.

Let
\[ S_k(x) := \sum_{p_1 \cdots p_k \leq x} \frac{1}{p_1 \cdots p_k} \]
\[ (x \geq 2), \]
where \( p_j \) denotes a prime number. It is a well known result of Mertens that
\[ S_1(x) = \log_2 x + c_1 + O\left(\frac{1}{\log x}\right) \]
\[ (x \geq 3), \]
with (see, e.g., [3], p. 18)
\[ c_1 := \gamma - \sum_p \left\{ \log \left(\frac{1}{1-1/p}\right) - \frac{1}{p} \right\} \approx 0.261497. \]

Here and in the sequel, \( \gamma \) is Euler’s constant, \( p \) stands for a prime number and \( \log_2 \) denotes the two-fold iterated logarithm. The number \( c_1 \) is called Mertens’ constant, also known as the Meissel-Mertens, or the Kronecker, or the Hadamard-La Vallée-Poussin constant.

In [1], [2], Popa used elementary techniques to derive similar asymptotic formulae in the cases \( k = 2 \) and \( 3 \), with a main term equal to a polynomial of degree \( k \) in \( \log_2 x \) and a remainder term \( \asymp (\log_2 x)^k / \log x \). In this note we investigate the general case. We define classically \( \Gamma \) as the Euler gamma function.

**Theorem 1.** Let \( k \geq 1 \). We have
\[ S_k(x) = P_k(\log_2 x) + O\left(\frac{(\log_2 x)^k}{\log x}\right) \]
\[ (x \geq 3), \]
where \( P_k(X) := \sum_{0 \leq j \leq k} \lambda_{j,k} X^j \), and
\[ \lambda_{j,k} := \sum_{m \leq k-j \leq m} \binom{k}{m,j,k-m-j}(c_1 - \gamma)^{k-m-j}\left(\frac{1}{\Gamma}\right)^{(m)}(1) \quad (0 \leq j \leq k). \]

**Proof.** Write \( P(s) := \sum_p 1/p^s \), so that we have
\[ P(s) = \log \zeta(s) - g(s), \quad g(s) := \sum_{m \geq 2} \frac{1}{m} \sum_p \frac{1}{p^ms} \]
in any simply connected zero and pole-free region of the zeta function where the series \( g(s) \) converges. (Here \( \log \zeta(s) \) is the branch that is real for real \( s > 1 \).) Moreover, for \( s + 1 \) in the same region, we have
\[ P(s + 1) = \log(1/s) + h(s), \]
with \( h(s) = \log \{s\zeta(s+1)\} - g(s + 1) \) and where \( \log(1/s) \) is understood as the principal branch. The function \( h(s) \) is clearly holomorphic in a disk around \( s = 0 \).

Now, for any \( c > 0 \), we have
\[ S_k(x) = \frac{1}{2\pi i} \int_{c+i\mathbb{R}} P(s+1)^k x^s \frac{ds}{s} \quad (x \in \mathbb{R}^+ \setminus \mathbb{N}). \]

By following, *mutatis mutandis*, the argument of the Selberg-Delange method (see [3], ch. II.5 & II.6) we readily obtain
\[ S_k(x) = \frac{1}{2\pi i} \int_{\mathcal{H}} \left\{ \log \left(\frac{1}{s}\right) + h(0)\right\}^k x^s \frac{ds}{s} + O\left(\frac{(\log_2 x)^k}{\log x}\right) \]
\[ (x \geq 2), \]
where \( \mathcal{H} \) is a Hankel contour around \( \mathbb{R}^- \), positively oriented.
We also observe that, by (1), we have
\[ h(0) = - \sum_p \left\{ \log \left( \frac{1}{1 - 1/p} \right) - \frac{1}{p} \right\} = c_1 - \gamma. \]

It remains to compute
\[ I_m(x) := \frac{1}{2\pi i} \int_{C} \left\{ \log \frac{1}{s} \right\}^m x^s \frac{ds}{s} \quad (m \geq 0). \]

To this end, we consider Hankel’s formula (see, e.g., [3], th. II.0.17)
\[ \frac{1}{2\pi i} \int_{C} x^s \frac{ds}{s^{1+\varepsilon}} = \frac{(\log x)^z}{\Gamma(z+1)} \quad (z \in \mathbb{C}) \]
and derive
\[ I_m(x) = \sum_{0 \leq j \leq m} \binom{m}{j} (\log_2 x)^j \left( \frac{1}{\Gamma} \right)^{(m-j)}(1). \]

Rearranging the terms, we arrive at the announced formula for \( P_k(X) \).

Specialization. Noting that \((1/\Gamma)'(1) = \gamma\), \((1/\Gamma)''(1) = \gamma^2 - \frac{1}{6}\pi^2\), \((1/\Gamma)^{(3)}(1) = 2\zeta(3) - \frac{1}{2}\pi^2\gamma + \gamma^3\), \((1/\Gamma)^{(4)}(1) = \frac{1}{60}\pi^4 + 8\gamma\zeta(3) + 3\pi^2\gamma^2 + \gamma^4\), as may be deduced from classical formulae for the logarithmic derivative of the Euler function (see, e.g., [3], chap. II.0), we find
\[
\begin{align*}
P_1(X) &= X + c_1, \\
P_2(X) &= (X + c_1)^2 - \frac{1}{6}\pi^2, \\
P_3(X) &= (X + c_1)^3 - \frac{1}{2}\pi^2(X + c_1) + 2\zeta(3), \\
P_4(X) &= (X + c_1)^4 - \pi^2(X + c_1)^2 + 8\zeta(3)(X + c_1) + \frac{1}{60}\pi^4.
\end{align*}
\]

Remark. By retaining, in the integrand of (2), the first \( N + 1 \) terms of the Taylor expansion of \( h(s) \) at the origin, the above method readily yields, for arbitrary integer \( N \geq 0 \), an asymptotic formula of the type
\[ S_k(x) = \sum_{0 \leq j \leq N} \frac{P_{j,k}(\log_2 x)}{(\log x)^j} + O \left( \frac{(\log_2 x)^k}{(\log x)^{k+N+1}} \right) \]
where \( P_{j,k} \) is an explicit polynomial of degree \( k \).

Acknowledgement. The author wishes to express warm thanks to Dumitru Popa for sending his works on the subject and for subsequent interesting conversations on the problem.

References