The distribution of integers with
at least two divisors in a short interval

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1 Introduction

Whereas, in usual cases, sieving by a set of primes may be fairly well controlled, through Buchstab’s
identity, sieving by a set of integers is a much more complicated task. However, some fairly precise
results are known in the case where the set of integers is an interval. We refer to the recent work [1]
of the first author for specific statements and references.

Define

\[ \tau(n; y, z) := |\{d|n : y < d \leq z\}|, \]
\[ H(x, y, z) := |\{n \leq x : \tau(n; y, z) \geq 1\}|, \]
\[ H_r(x, y, z) := |\{n \leq x : \tau(n; y, z) = r\}|, \]
\[ H^*_2(x, y, z) := |\{n \leq x : \tau(n; y, z) \geq 2\}| = \sum_{r \geq 2} H_r(x, y, z). \]

Thus, the numbers \( H_r(x, y, z) (r \geq 1) \) describe the local laws of the function \( \tau(n; y, z) \). When \( y \) and \( z \) are close, it is expected that, if an integer has at least a divisor in \((y, z]\), then it usually has
exactly one, in other words

\[ H(x, y, z) \sim H_1(x, y, z). \tag{1.1} \]

In this paper, we address the problem of determining the exact range of validity of such behavior.
In other words, we search for a necessary and sufficient condition so that \( H^*_2(x, y, z) = o(H(x, y, z)) \)
as \( x \) and \( y \) tend to infinity. We show below that (1.1) holds if and only if

\[ \lfloor y \rfloor + 1 \leq z < y + \frac{y}{(\log y) \log 4 - 1 + o(1)} \quad (y \to \infty). \]

As with the results in [1], the ratios \( H(x, y, z)/x \) and \( H_r(x, y, z)/x \) are weakly dependent on \( x \) when \( x \geq y^2 \). We take pains to prove results which are valid throughout the range \( 10 \leq y \leq \sqrt{x} \), since many interesting applications require bounds for \( H(x, y, z) \) and \( H_r(x, y, z) \) when \( y \approx \sqrt{x} \); see e.g. §1 of [1] and Ch. 2 of [4] for some examples.

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As shown in [6], for given $y$, the threshold for the behavior of the function $H(x, y, z)$ lies near the critical value

$$z = z_0(y) := y \exp\{(\log y)^{1-\log 4}\} \approx y + y/(\log y)^{\log 4 - 1}.$$  

We concentrate on the case $z_0(y) \leq z \leq 2y$. Define

$$z = e^\eta y, \quad \eta = (\log y)^{-\beta}, \quad \beta = \log 4 - 1 - \Xi/\sqrt{\log_2 y}, \quad \lambda = \frac{1 + \beta}{\log 2},$$

$$Q(w) = \int_1^w \log t \, dt = w \log w - w + 1.$$  \hspace{1cm} (1.2)

Here $\log_k^n$ denotes the $k$th iterate of the logarithm.

With the above notation, we have

$$\log(\frac{z}{y}) = e^{\Xi} \sqrt{\frac{\log_2 y}{\log y}}, \quad \log\{\frac{z}{z_0(y)}\} = e^{\Xi} \sqrt{\frac{\log_2 y - 1}{\log y}},$$

so

$$0 \leq \Xi \leq (\log 4 - 1) \sqrt{\log_2 y} + \frac{\log_2 2}{\sqrt{\log_2 y}},$$  \hspace{1cm} (1.3)

$$\frac{|\log_2 2|}{\log_2 y} \leq \beta \leq \log 4 - 1,$$  \hspace{1cm} (1.4)

$$\frac{1}{\log 2} + \frac{|\log_2 2|/\log 2}{\log_2 y} \leq \lambda \leq 2.$$  \hspace{1cm} (1.5)

From Theorem 1 of [1], we know that, uniformly in $10 \leq y \leq \sqrt{x}$, $z_0(y) \leq z \leq 2y$,

$$H(x, y, z) \asymp \frac{\beta x}{(\Xi + 1)(\log y)^{Q(\lambda)}}.$$  \hspace{1cm} (1.6)

By Theorems 5 and 6 of [1], for any $c > 0$ and uniformly in $y_0(r) \leq y \leq x^{1/2 - c}$, $z_0(y) \leq z \leq 2y$ for a suitable constant $y_0(r)$, we have

$$\frac{H_1(x, y, z)}{H(x, y, z)} \asymp_c 1,$$

$$\frac{\Xi + 1}{\sqrt{\log_2 y}} \ll_{r,c} \frac{H_r(x, y, z)}{H(x, y, z)} \leq 1 \quad (r \geq 2).$$  \hspace{1cm} (1.7)

When $0 \leq \Xi \leq o(\sqrt{\log_2 y})$ and $r \geq 2$, the upper and lower bounds above for $H_r(x, y, z)$ have different orders. We show in this paper that the lower bound represents the correct order of magnitude.

**Theorem 1.** Uniformly in $10 \leq y \leq \sqrt{x}$, $z_0(y) \leq z \leq 2y$, we have

$$\frac{H_2^r(x, y, z)}{H(x, y, z)} \ll \frac{\Xi + 1}{\sqrt{\log_2 y}},$$

where $\Xi = \Xi(y, z)$ is defined as in (1.2) and therefore satisfies (1.3).
Corollary 2. Let $r \geq 2$ and $c > 0$. There exists a constant $y_0(r, c)$ such that, uniformly for $y_0(r, c) \leq y \leq x^{1/2-c}$, $z_0(y) \leq z \leq 2y$, we have

$$\frac{H_r(x, y, z)}{H(x, y, z)} \preceq_{r, c} \frac{\Xi + 1}{\sqrt{\log y}}$$

Theorem 1 tells us that $H^*_2(x, y, z) = o(H(x, y, z))$ whenever $z \geq z_0(y)$ and $\Xi = o(\sqrt{\log y})$. It is a simple matter to adapt the proofs given in [5] to show that this latter relation persists in the range $\lfloor y \rfloor + 1 \leq z \leq z_0(y)$. We thus obtain the following statement.

Corollary 3. If $y \to \infty$, $y \leq \sqrt{x}$, and $\lfloor y \rfloor + 1 \leq z \leq y + y(\log y)^{1-\log 4+o(1)}$, we have

$$H_1(x, y, z) \sim H(x, y, z).$$

Since we know from (1.7) that $H^*_2(x, y, z) \gg_\varepsilon H(x, y, z)$ when $\beta \leq \log 4 - 1 - \varepsilon$ for any fixed $\varepsilon > 0$ we have therefore completely answered the question raised at the beginning of this introduction concerning the exact validity range for the asymptotic formula (1.1). This may be viewed as a complement to a theorem of Hall (see [3], ch. 7; following a note mentioned by Hall in private correspondence, we slightly modify the statement) according to which

$$H(x, y, z) \sim F(-\Xi) \sum_{r \geq 1} r H_r(x, y, z) = F(-\Xi) \sum_{n \leq x} \tau(n; y, z)$$

in the range $\Xi = o(\log y)^{1/6}$, $x > \exp\{\log z \log z\}$ with

$$F(\xi) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\xi/\log 4} e^{-u^2} \, du.$$  

It is likely that (1.8) still holds in the range $(\log y)^{1/6} \ll \Xi \leq o(\sqrt{\log y})$.

2 Auxiliary estimates

In the sequel, unless otherwise indicated, constants implied by Landau’s $O$ and Vinogradov’s $\ll$ symbols are absolute and effective. Numerical values of reasonable size could easily be given if needed.

Let $m$ be a positive integer. We denote by $P^-(m)$ the smallest, and by $P^+(m)$ the largest, prime factor of $m$, with the convention that $P^-(1) = \infty$, $P^+(1) = 1$. We write $\omega(m)$ for the number of distinct prime factors of $m$ and $\Omega(m)$ for the number of prime power divisors of $m$. We further define

$$\omega(m; t, u) = \sum_{\nu^e \parallel m, \nu \leq \nu^e \parallel m} 1,$$

$$\Omega(m; t, u) = \sum_{\nu^e \parallel m, \nu \leq \nu^e \parallel m} \nu,$$

$$\Omega(m; t) = \Omega(m; 2, t),$$

$$\Omega(m) = \Omega(m; 2, m).$$

Here and in the sequel, the letter $p$ denotes a prime number. Also, we let $\mathcal{P}(u,v)$ denote the set of integers all of whose prime factors are in $[u,v]$ and write $\mathcal{P}^*(u,v)$ for the set of squarefree members of $\mathcal{P}(u,v)$. By convention, $1 \in \mathcal{P}^*(u,v)$.  

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Lemma 2.1. There is an absolute constant $C > 0$ so that for $\frac{3}{2} \leq u < v$, $v \geq e^4$, $0 \leq \alpha \leq 1/\log v$, we have

$$\sum_{\substack{m \in \mathcal{P}(u,v) \\ \omega(m)=k}} \frac{1}{m^{1-\alpha}} \ll \frac{(\log_2 v - \log_2 u + C)^k}{k!}.$$ 

Proof. For a prime $p \leq v$, we have $p^\alpha \ll 1 + 2\alpha \log p$, thus the sum in question is

$$\ll \frac{1}{k!} \left( \sum_{u < p \leq v} \frac{1}{p^{1-\alpha}} + \frac{1}{p^{2-2\alpha}} + \cdots \right)^k \ll \frac{(\log_2 v - \log_2 u + O(1))^k}{k!}.$$ 

We note incidentally that a similar lower bound is available when $u$ and $v$ are not too close. See for instance Lemma III.13 of [2].

Lemma 2.2. Uniformly for $u \geq 10$, $0 \leq k \leq 2.9 \log_2 u$, and $0 \leq \alpha \leq 1/(100 \log u)$, we have

$$\sum_{\substack{P^+(m) \leq u \\ \prod(m)=k}} \frac{1}{m^{1-\alpha}} \ll \frac{(\log_2 u)^k}{k!}.$$ 

Proof. We follow the proof of Theorem 08 of [4]. Let $w$ be a complex number with $|w| \leq \frac{29}{10}$. If $p$ is prime and $3 \leq p \leq u$, then $|w/p^{1-\alpha}| \leq \frac{99}{100}$ and $p^\alpha \ll 1 + 2\alpha \log p$. Thus,

$$S(w) := \sum_{P^+(m) \leq u} \frac{w^{\prod(m)}}{m^{1-\alpha}} = \left(1 - \frac{1}{2^{1-\alpha}} \right)^{-1} \prod_{3 \leq p \leq u} \left(1 - \frac{w}{p^{1-\alpha}} \right)^{-1} \ll e^{(\Re w) \log_2 u}.$$ 

Put $r := k/\log_2 u$. By Cauchy’s formula and Stirling’s formula,

$$\sum_{\substack{P^+(m) \leq u \\ \prod(m)=k}} \frac{1}{m^{1-\alpha}} = \frac{1}{2\pi r^k} \int_{-\pi}^{\pi} e^{-ik\vartheta} S(re^{i\vartheta}) d\vartheta \ll \frac{(\log_2 u)^k}{r^k} \int_{-\pi}^{\pi} e^{r \cos \vartheta} d\vartheta \ll \frac{(\log_2 u)^k}{k!}.$$ 

Lemma 2.3. Suppose $z$ is large, $0 \leq a + b \leq \frac{5}{2} \log_2 z$ and

$$\exp\{(\log x)^{9/10}\} \leq w \leq z \leq x, \quad xz^{-1/(10 \log_2 z)} \leq Y \leq x.$$ 

The number of integers $n$ with $x - Y < n < x$, $\Omega(n;w) = a$ and $\omega(n;w,z) = \Omega(n;w,z) = b$, is

$$\ll Y \frac{\{\log_2 w\}^a}{a!} \frac{(b+1)\{\log_2 z - \log_2 w + C\}^b}{b!},$$

where $C$ is a positive absolute constant.
Proof. There are \( \ll x^{9/10} \) integers with \( n \leq x^{9/10} \) or \( 2^j|n \) with \( 2^j \geq x^{1/10} \). For other \( n \), write \( n = rst \), where \( P^+(r) \leq w, s \in \mathcal{P}^+(w,z) \) and \( P^-(t) > z \). Here \( \Omega(r) = a \) and \( \omega(s) = b \). We have either \( t = 1 \) or \( t > z \). In the latter case \( x/rs > z \), whence \( Y/rs > \sqrt{z} \). We may therefore apply a standard sieve estimate to bound, for given \( r \) and \( s \), the number of \( t \) by

\[
\ll \frac{Y}{rs \log z},
\]

By Lemmas 2.1 and 2.2,

\[
\sum_{r,s} \frac{1}{rs} \ll \frac{(\log_2 w)^a(\log_2 z - \log_2 w + C)^b}{a!b!}.
\]

If \( t = 1 \), then we may assume \( a + b \geq 1 \). Set \( p = P^+(n) \). If \( b \geq 1 \), then \( p|s \) and we put \( r_1 := r \) and \( s_1 := s/p \). Otherwise, let \( r_1 := r/p \) and \( s_1 := s = 1 \). Let \( A := \Omega(r_1) \) and \( B := \omega(s_1) \), so that \( A + B = a + b - 1 \) in all circumstances. We have

\[
p \geq x^{1/2\Omega(n)} \geq x^{1/5\log z} \geq (x/Y)^2.
\]

Define the non-negative integer \( h \) by \( x^{e^{-h-1}} < p \leq x^{e^{-h}} \). By the Brun-Titchmarsh theorem, we see that, for each given \( h \), \( r_1 \) and \( s_1 \), the number of \( p \) is \( \ll Ye^h/(r_1s_1 \log z) \). Set \( \alpha := 0 \) if \( h = 0 \) and \( \alpha := e^h/(100 \log z) \) otherwise. For \( h \geq 1 \), we have \( r_1s_1 > x^{3/4}z^{-1/e} > \sqrt{z} \). Therefore, for \( h \geq 0 \),

\[
\frac{1}{r_1s_1} \ll \frac{z^{a/2}/(r_1s_1)^{1-a}}{(r_1s_1)^{1-a}}.
\]

Now, Lemmas 2.1 and 2.2 imply that

\[
\sum_{r_1,s_1} \frac{1}{(r_1s_1)^{1-a}} \ll \frac{(\log_2 w)^a(\log_2 z - \log_2 w + C)^B}{A!B!} \ll \frac{(b + 1)(\log_2 w)^a(\log_2 z - \log_2 w + C)^b}{a!b!},
\]

where we used the fact that \( a \ll \log_2 w \). Summing over all \( h \), we derive that the number of those integers \( n > x^{9/10} \) satisfying the conditions of the statement is

\[
\ll \frac{Y}{\log z} (b + 1) \frac{(\log_2 w)^a(\log_2 z - \log_2 w + C)^b}{a!b!}.
\]

Since \( a!b! \leq (3 \log_2 z)^{3\log_2 z} \), this last expression is \( > x^{9/10} \). This completes the proof. \( \square \)

Our final lemma is a special case of a theorem of Shiu (Theorem 03 of [4]).

Lemma 2.4. Let \( f \) be a multiplicative function such that \( 0 \leq f(n) \leq 1 \) for all \( n \). Then, for all \( x \), \( Y \) with \( 1 < \sqrt{x} \leq Y \leq x \), we have

\[
\sum_{x-Y<n \leq x} f(n) \ll \frac{Y}{\log x} \exp \left\{ \sum_{p \leq x} \frac{f(p)}{p} \right\}.
\]
3 Decomposition and outline of the proof

Throughout, $\varepsilon$ will denote a very small positive constant. Note that Theorem 1 holds trivially for $\beta \leq \log 4 - 1 - \varepsilon$ since we then have $1 \ll \Xi/\log y$ and of course $H_4^*(x, y, z) \leq H(x, y, z)$. We may henceforth assume that
\[
\log 4 - 1 - \varepsilon \leq \beta \leq \log 4 - 1.
\] (3.1)

Let
\[ K := \lfloor \lambda \log_2 z \rfloor, \]
so that $(2 - \frac{3}{2} \varepsilon) \log_2 z \leq K \leq 2 \log_2 z$. In light of (1.6), Theorem 1 reduces to
\[
H_4^*(x, y, z) \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \tag{3.2}
\]

At this stage, we notice for further reference that, by Stirling’s formula, for $k \leq K$ we have
\[
\frac{\eta(2 \log_2 z)^k}{k!(\log z)^2} \leq \frac{\eta(2 \log_2 z)^K}{K!(\log z)^2} \times \frac{1}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \tag{3.3}
\]

Let $\mathcal{H}$ denote the set of integers $n \leq x$ with $\tau(n; y, z) \geq 2$. We count separately the integers $n \in \mathcal{H}$ lying in 6 classes. In these definitions, we write $k = \Pi(n; z)$, $b = K - k$ and for brevity we put $z_h = z^{(\beta - \varepsilon)}$. Let
\[
K_0 := (2 - 3\varepsilon) \log_2 z
\]
and define
\[
\mathcal{N}_0 := \{n \in \mathcal{H} : n \leq x/\log z \text{ or } \exists d > \log z : d^2 | n\},
\]
\[
\mathcal{N}_1 := \{n \in \mathcal{H} \setminus \mathcal{N}_0 : k \notin (K_0, K]\},
\]
\[
\mathcal{N}_2 := \bigcup_{1 \leq h \leq 5 \varepsilon \log_2 z} \mathcal{N}_{2,h},
\]
with $\mathcal{N}_{2,h} := \{n \in \mathcal{H} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1) : \Pi(n, z_h, z) \leq \frac{19}{10} h - \frac{1}{100} b\}.
\]

For integers $n \in \mathcal{N}_2$, we will only use the fact that $\tau(n; y, z) \geq 1$. Integers in other classes do not have too many small prime factors and it is sufficient to count pairs of divisors $d_1, d_2$ of $n$ in $(y, z)$. For each such pair, write $v = (d_1, d_2)$, $d_1 = vf_1$, $d_2 = vf_2$, $n = f_1 f_2 vv$ and assume $f_1 < f_2$. Let
\[
F_1 = \Pi(f_1), \quad F_2 = \Pi(f_2), \quad V = \Pi(v), \quad U = \Pi(u, z), \tag{3.4}
\]
and
\[
Z := \exp\{(\log z)^{1-4\varepsilon}\}. \tag{3.5}
\]

For further reference, we note that if $n \notin \mathcal{N}_0$ and $h \leq 5 \varepsilon \log_2 z$, then
\[
\Pi(n; z_h, z) = \omega(n; z_h, z).
\]

Now we define $\mathcal{H}^* := \mathcal{H} \setminus \mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2$ and
\[
\mathcal{N}_3 := \{n \in \mathcal{H}^* : \min(u, f_2) \leq Z\},
\]
\[
\mathcal{N}_4 := \{n \in \mathcal{H}^* : \min(u, f_2) > z^{1/10}\},
\]
\[
\mathcal{N}_5 := \{n \in \mathcal{H}^* : Z < \min(u, f_2) \leq z^{1/10}\}.
\]
In the above decomposition, the main parts are $N_2$ and $N_5$. We expect $N_2$ to be small since, conditionally on $\Omega(n; z) = k$, the normal value of $\Omega(n; z_h, z)$ is $hk/\log_2 z > \frac{10}{100} h$. It is more difficult to see that $N_5$ is small too. This follows from the fact that we count integers in this set according to their number of factorizations in the form $n = uv f_1 f_2$ with $y < v f_1 < v f_2 < z$. Suppose for instance that $f_1, f_2 \leq z_j$. For $\Omega(n; z) = k$ and $\Omega(n; z_j, z) = G$, then, ignoring the given information on the localization of $v f_1$ and $v f_2$ in $(y, z]$, there are $4^{k-G} G^2 = 4^{k-2-G}$ such factorizations. Thus, larger $G$ means fewer factorizations. On probabilistic grounds, larger $G$ should also mean fewer factorizations when information on the localization of $v f_1$ and $v f_2$ is available.

We now briefly consider the cases of $N_0$ and $N_1$. Trivially,

$$|N_0| \leq \frac{x}{\log z} + \sum_{d > \log z} \frac{x}{d^2} \ll \frac{x}{(\log y)Q(\lambda)\sqrt{\log_2 y}},$$

since $Q(\lambda) \leq Q(2) = \log 4 - 1$ in the range under consideration.

By the argument on pages 40–41 of [4],

$$\sum_{n \leq x \atop \Pi(n; z) > K} 1 \ll \frac{x}{(\log y)Q(\lambda)\sqrt{\log_2 y}}.$$

Setting $t := 1 - \frac{3}{2} \varepsilon$, Lemma 2.4 gives

$$\sum_{n \leq x \atop \tau(n; y, z) \geq 1 \atop \Pi(n; z) \leq K_0} 1 \ll x (\log z)^2 - \beta -(2 - 3\varepsilon) \log t \ll x (\log y)^{-\beta - 2\varepsilon^2} \ll x (\log y)^{-Q(\lambda)-\varepsilon^2/2}.$$

Therefore,

$$|N_1| \ll \frac{x}{(\log y)Q(\lambda)\sqrt{\log_2 y}}.$$  \hfill (3.7)

In the next four sections, we show that

$$|N_j| \ll \frac{x}{(\log y)Q(\lambda)\sqrt{\log_2 y}} \quad (2 \leq j \leq 5).$$  \hfill (3.8)

Together with (3.6) and (3.7), this will complete the proof of Theorem 1.

### 4 Estimation of $|N_2|$  

We plainly have $|N_2| \leq \sum_h |N_{2,h}|$. For $1 \leq h \leq 5 \varepsilon \log_2 z$, the numbers $n \in N_{2,h}$ satisfy

$$\begin{cases} 
  x/\log z < n \leq x, \\
  k := \Omega(n; z) = K - b, \quad 0 \leq b \leq 3 \varepsilon \log_2 z, \\
  \Omega(n; z_h, z) \leq \frac{10}{100} h - \frac{1}{100} b,
\end{cases}$$

We note at the outset that $N_{2,h}$ is empty unless $h \geq b/190$.  

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Write \( n = du \) with \( y < d \leq z \) and \( u \leq x/y \). Let
\[
\Omega(d; z_h) = D_1, \quad \Omega(d; z_h, z) = D_2, \quad \Omega(u; z_h) = U_1, \quad \Omega(u; z_h, z) = U_2,
\]
so that \( D_1 + D_2 \geq 1, D_2 + U_2 \leq 1/10 h - 1/100 b \) and \( D_1 + D_2 + U_1 + U_2 = k \).

Fix \( k = K - b, h, D_1, D_2, U_1 \) and \( U_2 \). By Lemma 2.3 (with \( w = z_h, a = U_1, b = U_2 \)), the number of \( u \) is
\[
\ll \frac{x}{y \log z} \frac{(\log_2 z - h) U_1 (U_2 + 1)(h + C) U_2}{U_1! U_2!}.
\]
A second application of Lemma 2.3 yields that the number of \( d \) is
\[
\ll \frac{\eta y}{\log z} \frac{(\log_2 z - h) D_1}{D_1!} (D_2 + 1)(h + C) D_2 \frac{1}{D_2!}.
\]
Since \( D_2 + U_2 < 2h \), we have \( (h + C) U_2 + D_2 \leq e^{2C} h U_2 + D_2 \). Summing over \( D_1, D_2, U_1, U_2 \) with \( G = D_2 + U_2 \) fixed and using the binomial theorem, we find that the number of \( n \) in question is
\[
\ll \frac{\eta x}{(\log z)^2} (\log_2 z - h)^{k - G} h^G (G + 1)^2 \sum_{U_1 + D_1 = k - G} \sum_{D_2 + U_2 = G} \frac{1}{U_1! D_1! D_2! U_2!} \ll \frac{\eta x 2^k}{(\log z)^2} A(h, G),
\]
where
\[
A(h, G) = (G + 1)^2 \frac{(\log_2 z - h)^{k - G} h^G}{(k - G)! G!}.
\]
Since \( G + 1 \leq G_h := \lfloor 1/10 h \rfloor \), we have
\[
\frac{A(h, G + 1)}{A(h, G)} \geq \frac{h (k - G)}{(G + 1)(\log_2 z - h)} \geq \frac{k - 10 \varepsilon \log_2 z}{1.9 (1 - 5 \varepsilon) \log_2 z} > 21/20
\]
if \( \varepsilon \) is small enough. Next,
\[
A(h, G_h) \ll (G_h + 1)^2 \frac{(\log_2 z - h)^{k - G_h} (h h)^{G_h}}{k! (G_h/h)^{G_h}} \ll \frac{(h + 1)^2 (\log_2 z)^k}{k!} \left( \frac{20}{\ln 19} \right)^{10 h/10} e^{-h (k - G_h)/\log_2 z} \ll \frac{(\log_2 z)^k}{k!} e^{-h/500},
\]
since \( (k - G_h)/\log_2 z > 2 - 13 \varepsilon \) and \( 10/19 \log (20/19 e) < 2 - 1/400 \). Thus,
\[
\sum_{b/190 \leq h \leq 5 \varepsilon \log_2 z} \sum_{G \leq G_h} A(h, G) \ll \sum_{b/190 \leq h \leq 5 \varepsilon \log_2 z} A(h, G_h) \ll \frac{(\log_2 z)^k}{k!} e^{-b/95000}
\]
and so
\[
\sum_{n \in \Lambda_2 \in \mathfrak{M}(n; z) = k} 1 \ll \frac{\eta x (2 \log_2 z)^k}{(\log z)^2 k!} e^{-(K - k)/95000} \ll \frac{x e^{-(K - k)/95000}}{(\log y) Q(\lambda) \sqrt{\log_2 y}}
\]
by (3.3). Summing over the range \( K_0 \leq k \leq K \) furnishes the required estimate (3.8) for \( j = 2 \).
5 Estimation of $|N_3|$ 

All integers $n = f_1 f_2 v u$ counted in $N_3$ verify

$$
\begin{aligned}
\left\{ \begin{array}{l}
x \log z < n \leq x, \\
\Pi(n; z) \leq K, \\
y < v f_1 < v f_2 \leq z, \quad \min(u, f_2) \leq Z,
\end{array} \right.
\end{aligned}
$$

where $Z$ is defined in (3.5). This is all we shall use in bounding $|N_3|$.

Let $N_{3,1}$ be the subset corresponding to the condition $f_2 \leq Z$ and let $N_{3,2}$ comprise those $n \in N_3$ such that $u \leq Z$.

If $f_2 \leq Z$, then $v > z^{1/2}$ and $u > x/\{v Z^2 \log z\} > x^{1/3}$. For $\frac{1}{2} \leq t \leq 1$ we have

$$
|N_{3,1}| \leq \sum_{f_1, f_2, v, u} t^{\Pi(f_1 f_2 u v; z) - K}
= t^{-K} \sum_{f_1 \leq Z} t^{\Pi(f_1)} \sum_{f_2 < f_1 \leq e^2 f_1} t^{\Pi(f_2)} \sum_{y/f_1 < v \leq z/f_1} t^{\Pi(v)} \sum_{u \leq x/f_1 f_2} t^{\Pi(u; z)}.
$$

Apply Lemma 2.4 to the three innermost sums. The $u$-sum is

$$
\ll \frac{x}{f_1 f_2 v} (\log z)^{t-1} \leq \frac{x}{f_1 y} (\log z)^{t-1},
$$

and the $v$-sum is

$$
\ll \frac{\eta y}{f_1} (\log z)^{t-1}.
$$

The $f_2$-sum is $\ll \eta f_1 (\log f_1)^{t-1}$ if $f_1 > \eta^{-3}$ and otherwise is $\ll \eta f_1$ trivially (note that $\eta f_1 \gg 1$ follows from the fact that $(f_1 + 1)/f_1 \leq f_2/f_1 \leq e^0$). Next

$$
\sum_{f_1 \leq \eta^{-3}} \frac{1}{f_1} + \sum_{2 \leq f_1 \leq Z} \frac{\Pi(f_1)}{f_1} (\log f_1)^{t-1} \ll \log z + (\log z)^{\max_{j \leq \log z} e^{j(t-1)}} \sum_{f_1 \leq \exp\{e\}} t^{\Pi(f_1)/f_1}
$$

$$
\ll (\log z)(\log Z)^{2t-1}.
$$

Thus,

$$
|N_{3,1}| \ll x (\log x) (\log x)^E
$$

with $E = -2\beta - \lambda \log t + 2t - 2 + (2t - 1)(1 - 4\varepsilon)$. We select optimally $t := \frac{1}{2} \lambda/(1 - 2\varepsilon)$, and check that $t \geq \frac{1}{2}$ since $\lambda \geq 2 - \varepsilon/\log 2$. Then

$$
E = -Q(\lambda) + \lambda \log(1 - 2\varepsilon) + 4\varepsilon \leq -Q(\lambda) + (2 - \varepsilon/\log 2)(-2\varepsilon - 2\varepsilon^2) + 4\varepsilon
$$

$$
< -Q(\lambda) - \varepsilon^2.
$$

Next, we consider the case when $u \leq Z$. We observe that this implies

$$
\frac{1}{4} v Z^2 \leq vx \leq vn \log z = uf_1 v f_2 v \log z \leq Z z^2 \log z
$$

hence $v \leq 4Z \log z \leq Z^2$, and therefore

$$
\min(f_1, f_2) \geq Z^{1/2}.
$$
Also, \( z > x^{1/3} \) since \( x/\log z < n = uvf_1f_2 \leq Z^2 \). Thus, for \( \frac{1}{2} \leq t \leq 1 \), we have

\[
|N_{3,2}| \leq \sum_{v \leq z^2} t^\Pi(v) \sum_{u \leq xy/v^2} t^\Pi(u) \sum_{y/v < f_1 \leq z/v} t^\Pi(f_1) \sum_{y/v < f_2 \leq z/v} t^\Pi(f_2).
\]

The sums upon \( f_1 \) and \( f_2 \) are each \( \ll \eta y (\log z)^t \) and the \( u \)-sum is \( \ll \frac{xv}{y^2} (\log 2)^t \leq \frac{xv}{y^2} (\log 2v)^t \).

Thus, selecting the same value \( t := \frac{1}{4} \lambda/(1 - 2\varepsilon) \), we obtain

\[
|N_{3,2}| \ll t^{-K} x\eta^2 (\log z)^{2t-1} \sum_{v \leq z^2} \frac{t^\Pi(v) (\log 2v)^{t-1}}{v} \ll x(\log_2 z)(\log z)^E \ll x(\log_2 z)(\log z)^{-Q(\lambda)-\varepsilon^2}.
\]

This completes the proof of (3.8) with \( j = 3 \).

6 Estimation of \( |N_4| \)

We now consider those integers \( n = f_1f_2uv \) such that

\[
\begin{cases}
  x/\log z < n \leq x, \\
  k := \Omega(n; z) = K - b, \quad 0 \leq b \leq 3\varepsilon \log_2 z, \\
  y < vf_1 < vf_2 \leq z, \quad \min(u, f_2) > z^{1/10}.
\end{cases}
\]

With the notation (3.4), fix \( k, F_1, F_2, U \) and \( V \). Here \( u, f_1 \) and \( f_2 \) are all \( > \frac{1}{2} z^{1/10} \). By Lemma 2.3 (with \( w = z \)), for each triple \( f_1, f_2, v \) the number of \( u \) is

\[
\ll \frac{x}{f_1f_2v \log z} \frac{(\log z)^U}{U!}.
\]

Using Lemma 2.3 two more times, we obtain, for each \( v \),

\[
\sum_{y/v < f_1 \leq z/v} \frac{1}{f_1} \sum_{y/v < f_2 \leq z/v} \frac{1}{f_2} \ll \frac{\eta^2}{(\log z)^2} \frac{(\log z)^{F_1+F_2}}{F_1!F_2!}.
\]

Now, Lemma 2.2 gives

\[
\sum_{v} \frac{1}{v} \ll \frac{(\log_2 z)^V}{V!}.
\]
Gathering these estimates and using (3.3) yields

\[ |N_4| \ll \frac{x \eta^2}{(\log z)^3} \sum_{(2-3\varepsilon) \log_2 z \leq k \leq K} \sum_{F_1 + F_2 + U + V = k} \frac{(\log_2 z)^k}{F_1! F_2! U! V!} \]

\[ = \frac{x \eta^2}{(\log z)^3} \sum_{(2-3\varepsilon) \log_2 z \leq k \leq K} \frac{(2 \log_2 z)^k}{k!} \]

\[ \ll \frac{x}{(\log y)^Q(\lambda)} \sqrt{\log_2 y \log z} \ll \frac{x}{(\log y)^Q(\lambda)} \sqrt{\log_2 y} . \]

Thus (3.8) holds for \( j = 4 \).

7 Estimation of \( |N_5| \)

It is plainly sufficient to bound the number of those \( n = f_1 f_2 uv \) satisfying the following conditions

\[
\begin{align*}
    x / \log z &< n \leq x, \\
    k := \tilde{\Omega}(n; z) = K - b, &\quad 0 \leq b \leq 3\varepsilon \log_2 z, \\
    \tilde{\Omega}(n; z_h, z) > \frac{10}{100} h - \frac{1}{100} b &\quad (1 \leq h \leq 5\varepsilon \log_2 z) \\
    y < v f_1 < v f_2 &\leq z, \quad Z < \min(u, f_2) \leq z^{1/10}.
\end{align*}
\]

Define \( j \) by \( z_{j+2} < \min(u, f_2) \leq z_{j+1} \). We have \( 1 \leq j \leq 5\varepsilon \log_2 z \). Let \( N_{5,1} \) be the set of those \( n \) satisfying the above conditions with \( u \leq z_{j+1} \) and let \( N_{5,2} \) be the complementary set, for which \( f_2 \leq z_{j+1} \).

If \( u \leq z_{j+1} \), then \( v \leq (z^2 u \log z) / x \leq 4u \log z \leq z_j \) and \( f_2 > f_1 > z^{1/2} \). Recall notation (3.4) and write

\[ F_{11} := \tilde{\Omega}(f_1; z_j), \quad F_{12} := \Omega(f_1; z_j, z), \quad F_{21} := \tilde{\Omega}(f_2; z_j), \quad F_{22} := \Omega(f_2; z_j, z), \]

so that the initial condition upon \( \tilde{\Omega}(n; z_h, z) \) with \( h = j \) may be rewritten as

\[ F_{12} + F_{22} \geq G_j := \max(0, |\frac{10}{100} j - b/100|) \]

We count those \( n \) in a dyadic interval \( (X, 2X] \), where \( x / (2 \log z) \leq X \leq x \). Fix \( k, j, X, U, V, F_{rs} \) and apply Lemma 2.3 to sums over \( u, f_1, f_2 \). The number of \( n \) is question is

\[
\ll \sum_{v \leq z_j} \sum_{v X / z^2 \leq u \leq 2eX / y^2} \sum_{y / v < f_1 \leq z / v} \sum_{y / v < f_2 \leq z / v} \frac{1}{v} \ll \frac{\eta^2 X e^j}{(\log z)^3} \frac{(\log_2 z - j) U + F_{11} + F_{21} }{U! F_{11}! F_{21}!} (F_{12} + 1) (F_{22} + 1) \frac{(j + C) F_{12} + F_{22} }{F_{12}! F_{22}!} \sum_{v \leq z_j} \frac{1}{v} .
\]

Bounding the \( v \)-sum by Lemma 2.2, and summing over \( X, U, V, F_{rs} \) with \( F_{12} + F_{22} = G \) yields

\[ |N_{5,1}| \ll \frac{\eta^2 x}{(\log z)^3} \sum_{(2-3\varepsilon) \log_2 z \leq k \leq K} 4^k \sum_{1 \leq j \leq 5\varepsilon \log_2 z} \sum_{G_j \leq G \leq k} M(j, G), \]

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Therefore, and hence, by (3.3),

\[ G \]

Thus, for

\[ k \]

since

\[ R := \max_{G \geq 0} \{ (G + 1)^2 \left( \frac{99}{100} \right)^G \} \]

we have

\[
\sum_{1 \leq j \leq j_b} \sum_{G_j \leq G \leq k} M(j, G) \leq R \sum_{1 \leq j \leq j_b} e^j \sum_{0 \leq G \leq k \leq j} \frac{(\log_2 z - j)^{k-G}(j + G)^G}{2^G G!(k - G)!} \\
\leq \frac{1}{k!} \sum_{1 \leq j \leq j_b} e^j (\log_2 z - j + \frac{1}{2} C_b)^k \\
\leq \frac{(\log_2 z)^k}{k!} \sum_{1 \leq j \leq j_b} e^j (200 - j/2 k)/\log_2 z \\
\leq \frac{(\log_2 z)^k}{k!} e^{j/(200+2j_b)} \leq \frac{(\log_2 z)^k}{k!} e^{j/50}.
\]

When \( j > j_b \), then

\[ G_j \geq \frac{9}{5} (j + C) + \frac{1}{10} (j_b + C + 1) - \frac{1}{100} b - 1 \geq \frac{9}{5} (j + C) + 9 \geq 189. \]

Thus, for \( G \geq G_j \) we have

\[
\frac{M(j, G + 1)}{M(j, G)} = \left( \frac{G + 2}{G + 1} \right)^2 \frac{j + C}{2(G + 1)} \frac{k - G}{\log_2 z - j} \leq \frac{4}{7}.
\]

Therefore,

\[
\sum_{G_j \leq G \leq k} M(j, G) \ll M(j, G_j) \ll \frac{j^2 e^j (\log_2 z - j)^{k-G_j}(j k)^{G_j}}{2^{G_j} G_j!} \\
\ll \frac{j^2 e^j (\log_2 z)^k}{k!} e^{-j (k - G_j)/\log_2 z} \left( \frac{e j k}{2 G_j \log_2 z} \right)^{G_j} \ll \frac{(\log_2 z)^k}{k!} e^{-j/5},
\]

since \( k - G_j \geq (2 - 10 \varepsilon) \log_2 z \), \( e j k/(2 G_j \log_2 z) \leq \frac{2}{9} \), and \(-1 + \frac{10}{11} \log(\frac{2}{9} \varepsilon) < -\frac{1}{5} \). We conclude that

\[
\sum_{1 \leq j \leq \log_2 z} \sum_{G_j \leq G \leq k} M(j, G) \ll \frac{(\log_2 z)^k}{k!} e^{j/50} \quad (7.1)
\]

and hence, by (3.3),

\[
|N_{5,1}| \ll \frac{\eta^2 x}{(\log z)^3} \sum_{k \leq K} \frac{(2 \log_2 z)^k}{k!} 2K^{-b/2} \ll \frac{\eta^2 2K x}{(\log z)^3} \frac{(2 \log_2 z)^K}{K!} \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log z} \log y}.
\]

Now assume \( f_2 \leq z_{j+1} \). Then \( \min(u, v) > \sqrt{z} \). Fix \( F_1, F_2 \) and

\[ \Omega(v; z_j) = V_1, \quad \Omega(v; z_j, z) = V_2, \quad \Omega(u; z_j) = U_1, \quad \Omega(u; z_j, z) = U_2. \]
By Lemma 2.3, given $f_1, f_2$ and $v$, the number of $u$ is

$$\ll \frac{x}{f_1 f_2 v \log z} \frac{(\log_2 z - j)^{U_1} (U_2 + 1) (j + C)^{U_2}}{U_1! U_2!}.$$  

Applying Lemma 2.3 again, for each $f_1$ we have

$$\sum_{f_1 < f_2 \leq e^\rho f_1 \atop y/f_1 < v \leq z/f_1} \frac{1}{f_2 v} \ll \frac{\eta^2 e^j (V_2 + 1) (\log_2 z - j)^{V_1 + F_2} (j + C)^{V_2}}{(\log z)^2 V_1! V_2! F_1!}.$$  

By Lemma 2.2,

$$\sum_{f_1 \leq j} \frac{1}{f_1} \ll \frac{(\log z - j)^{F_1}}{F_1!}.$$  

Combine these estimates, and sum over $F_1, F_2, U_1, U_2, V_1, V_2$ with $V_2 + U_2 = G$. As in the estimation of $|N_{5,1}|$, sum over $k, j, G$ using (3.3) and (7.1). We obtain

$$|N_{5,2}| \ll \frac{\eta^2 x}{(\log z)^3} \sum_{2 \leq k \leq K} 4^k \sum_{1 \leq j \leq \log_2 z} \sum_{G_j < G \leq k} M(j, G) \ll \frac{x}{(\log y) Q(\lambda) \sqrt{\log_2 y}}.$$  

References


